ISSN: 3033-3652

Journal of Physical Mathematics & its Applications

SCIENTIFIC Research and Community

Review Article

Open Access

Existence and Uniqueness of the Solution of a Mixed Problem for a Parabolic Equation Under Nonconventional Boundary Conditions

Yu. A. Mammadov and H.I. Ahmadov*

Department of Equations of Mathematical Physics, Faculty of Applied Mathematics and Cybernetics, Baku State University, st. Z. Khalilov 23, AZ-1148, Baku, Azerbaijan

ABSTRACT

Our research focuses on examining a mixed problem associated with a second-order parabolic equation that features temporal mixing and variable coefficients, subject to non-local and non-self-adjoint boundary conditions. We establish the problem's unique solvability by imposing specific conditions on the provided data, utilizing a combination of residue and contour integral methods. Additionally, our study produces an explicit analytical solution for addressing the problem at hand.

***Corresponding author**

H.I. Ahmadov, Department of Equations of Mathematical Physics, Faculty of Applied Mathematics and Cybernetics, Baku State University, st. Z. Khalilov 23, AZ-1148, Baku, Azerbaijan.

Received: July 01, 2024; **Accepted:** July 04, 2024; **Published:** July 15, 2024

Keywords: Parabolic Equation, Time Shift, Mixed Problem, Residue Method, Contour Integral Method

Introduction

Well know that the problem of so-called nonlocal boundary conditions are the included and developed by A.A. Samarsky and A.V. Bitsadze and plays important role of the theory differential equations and equations of mathematical physics, yet [1].

Presently, one of the important problem is the study of problems with various types of nonlocal boundary conditions, because multipoint boundary value problems for ordinary differential equation have many applications in the modeling and analysis of problems arising in electric power grids, electric railway systems, telecommunication lines, as well as in chemistry and analysis of kinetic reaction problems. They have been intensively studied in Refs. [2-8].

However, there are only a few works devoted to non-stationary problems along with multipoint boundary conditions, for example, [9-13]. The work deals with three-point boundary conditions subject to the nonlinear parabolic Cauchy problem [8].

Another side same problem also studied in Refs. [14-17]. In work [14] studied a semi linear parabolic equation in 1D along with nonlocal boundary conditions studied. The value at each boundary point is associated with the value at an interior point of the domain, which is known as a four-point boundary condition. First, the solvability of a steady-state problem is addressed and a constructive algorithm for finding a solution is proposed.

A method for regularizing boundary value problems for a parabolic equation was developed in [15]. A singularly perturbed boundary value problem on semiaxis is considered in the case of a simple rational turning point. To prove the asymptotic convergence of the series, the maximum principle is used.

Must be noted that in [16] is devoted to the fundamental problem of studying investigating the solvability of initial boundary value problems for a quasi-linear pseudo-parabolic equation of fractional order with a sufficiently smooth boundary. The difference between the studied problems is that the boundary conditions are set in the form of a nonlinear boundary condition with a fractional differentiation operator. The main result of this work is establishing the local or global solvability of stated problems, depending on the parameters of the equation. The Galerkin method is used to prove the existence of a quasi-linear pseudo-parabolic equation's weak solution in a bounded domain. Using Sobolev embedding theorems, a priori estimates of the solution are obtained. A priori estimates and the Rellich–Kondrashov theorem are used to prove the existence of the desired solutions to the considered boundary value problems.

In Ref. [17] considered an inverse problem of time fractional parabolic partial differential equations with the nonlocal boundary condition. Where Dirichlet-measured output data are used to distinguish the unknown coefficient. A finite difference scheme is constructed and a numerical approximation is made. Examples and numerical experiments, such as man-made noise, are provided to show the stability and efficiency of this numerical method.

The paper deals with a mixed problem for a heat-conductivity equation with time shift in nonlocal and not self-adjoint boundary conditions [18]. Unique solvability is proved under minimum conditions on the initial data and an explicit representation for solving the problem is obtained.

The papers [19,20] consider mixed problems for a parabolic type equation with constant coefficients under homogeneous and inhomogeneous boundary conditions with time shift in some of them. And also, under certain conditions on the data, by combining the residue and contour integral method [21,22] the unique solvability is proved and integral representations for solving the stated problems are obtained.

Unlike numerous known works devoted to problems for partial equations including parabolic ones, whereon the time deviatin is in the equation or instead of boundary ones there are functional conditions ([23]-[28] etc.), we consider the problem where the time shift of the desired function occurs in boundary conditions

Taking these points into account, it can be argued that the study of a mixed problem associated with a second-order parabolic equation characterized by temporary mixing and variable coefficients subject to non-local and non-self-adjoint boundary conditions is one the important problems. and an important problem in mathematical physics.

Problem statement:
Let

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u(x,t)=a(x)u_{xx}+b(x)u_x+c(x)u-u_t,$$

$$l_ju(x,t)=u(x,t+(1-j)\omega)+\delta_ju(1-x,t+j\omega),\ j=0,1$$

$$l_ju(x,t)=\alpha_{j-2}u_x^{(j-2)}(x,t)+\beta_{j-2}u_x^{(j-2)}(1-x,t),\ j=2,3$$

where $a(x),b(x),c(x)$ are the known coefficients and are real functions, $\omega,\delta_j,\alpha_j,\ \beta_j\ (j=0,1)$ are real constants, $\omega>0,\delta_0.\delta_1\neq 0$

In the half-plane $\Pi=\{(x,t):0<x<1,t>0\}$ we consider the following mixed problem

$$Lu(x,t)=0,\qquad (x,t)\in\Pi,\qquad (1)$$

$$u(x,0)=\varphi(x),\qquad 0<x<1,\ ,\qquad (2)$$

$$l_ju(x,t)\big|_{x=0}=0,\qquad t>0,\ \ j=0,1,\qquad (3)$$

$$l_ju(x,t)\big|_{x=0}=0, 0<t\le\omega, j=2,3.\qquad (4)$$

The solution of problem (1)-(4) is the function $u(x,t)$, satisfying the following conditions:

1) $u(x,t)\in C^{2,1}(\Pi)\cap C(0<x<t,\ t\ge 0)$;

$\int\limits_0^t u(x,\tau)d\tau\in C(0\le x\le 1,\ \ t\ge 0)$;

2) $l_ju(x,t)\in C(0\le x<1,\ t>0),\ \ j=0,1$;

3) $l_ju(x,t)\in C(0\le x<1,\ 0<t\le\omega),\ \ j=2,3$;

4) $u(x,t)$ satisfies the equalities (1)-(4) in the usual sense.

The uniqueness of the solution: The problem

$$L\left(\frac{d}{dx},\mu^2\right)y(x,\mu)=0,\qquad (5)$$

$$l_jy(x,\mu)\big|_{x=0}=0, j=2,3,\qquad (6)$$

is a said to be the first spectral problem with a complex parameter μ. Here

$$L\left(\frac{d}{dx},\mu^2\right)y(x,\mu)=a(x)y''+b(x)y'+c(x)y-\mu^2y.$$

It is known [21] that fundamental systems of particular solutions of equation (5) have the asymptotic of representation the form

$$y(x,\mu)=\left[B(x)+O\left(\frac{1}{\mu}\right)\right]\exp\left(\int\limits_0^x\left(e^{\mu w(\xi)}-\frac{b(\xi)}{2a(\xi)}\right)d\xi\right),\qquad (7)$$

where

$w(x)$ is a diagonal matrix of the following form

$$w(x)=\begin{pmatrix}\frac{1}{\sqrt{a(x)}} & 0\\ 0 & -\frac{1}{\sqrt{a(x)}}\end{pmatrix}$$

It is known [29,30] that if $a(0)\alpha_0\beta_1+a(1)\beta_0\alpha_1\neq 0$,

then for all complex values μ, where $\mu\neq\mu_\nu$,

$$\mu_\nu=\left[\int\limits_0^1\frac{dx}{\sqrt{a(x)}}\right]^{-1}\left(\ln_0\frac{1}{2}\left(A\pm\sqrt{A^2-4}\right)+2\pi\nu i\right)+O\left(\frac{1}{\nu}\right),\nu\to\infty\qquad (8)$$

$$A=\left[a(0)\alpha_0\beta_1+a(1)\beta_0\alpha_1\right]^{-1}\left(2\alpha_0\alpha_1\exp\left(\int\limits_0^1\frac{b(x)}{2a(x)}dx\right)+2\beta_0\beta_1\right)$$

there exists the green function $G_1(x,\xi,\mu)$ зof problem (5), (6), that is analytic for $\mu\neq\mu_1$. By S we denote the set of eigenvalues μ_ν , i.e. $S=\{\mu_\nu:\nu=1,2,\ldots\}$

Enumerating the points μ_ν (ν =1,2...) form S in ascending order of their modules taking into account their multiplicity, we have $|\mu_1|\le|\mu_2|\le\ldots,\ \mu_\nu$. We denote the multiplicity of the eigenvalue μ_ν by χ_ν . It is clear that $|\mu_\nu|\to\infty(\nu\to\infty)$, there exist such $h>0,\delta>0$, that

$$-h<Re\,\mu_\nu<h,\qquad |\mu_{\nu+1}-\mu_\nu|>2\delta,\quad (\nu=1,2,3,4\ldots)\quad (9)$$

From the Green function $G_1(x,\xi,\mu)$ the following estimation holds

$$\left|\frac{\partial^kG_1(x,\xi,\mu)}{\partial x^k}\right|\le C_0|\mu|^{k-1},(k=0,1,2),\qquad (10)$$

$C_0>0.$

If $f(x)\in C[0,1]$, then

$$L\left(\frac{d}{dx},\mu^2\right)\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=-f(x),\qquad (11)$$

$$l_j G_1(x,\xi,\mu)\big|_{x=0}=0, j=2,3.$$

Let $f(x)$ be from the domain of definition of the operator of the first spectral problem i.e.

$$f(x)\in C^2[0,1],\quad l_j f\big|_{x=0}=0, j=2,3.$$

Then we have the equality

$$\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=\frac{f(x)}{\mu^2}+\frac{1}{\mu^2}\int_0^1 G_1(x,\xi,\mu)\times$$

$$\times\left[a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)\right]d\xi.\quad (12)$$

Let $C>0, r>0$ be some numbers, z be a complex variable. Denote by $\Im_c=\{z:\ \operatorname{Re} z^2=c\}$ a hyperbola with the branches $\Im_c^{\pm}=\{z:\operatorname{Re} z^2=c,\pm\operatorname{Re} z>0\}$, by $\Omega_r=\{z:|z|=r\}$ a circle, $\Omega_r(\theta_1,\theta_2)$ is an area of the circle Ω_r enclosed between the rays $z=\sigma e^{i\theta_j}$ $(0\le\sigma<\infty,\ i=\sqrt{-1}, j=1,2)$.

Note that the arcs $\{z:|z|=r,\operatorname{Re} z^2\ge c,\operatorname{Re} z<0\}$ and $\{z:|z|=r,\operatorname{Re} z^2\le c,\operatorname{Im} z<0\}$ connecting the branches and the sides of the hyperbola $\Im_c$ in our denotations will be

$$\Omega_r(-\theta_{c,r},\theta_{c,r}),\Omega_r(\theta_{c,r},-\theta_{c,r}+\pi),\Omega_r(-\theta_{c,r}+\pi,\theta_{c,r}+\pi),$$

$$\Omega_r(\theta_{c,r}+\pi,-\theta_{c,r}+2\pi),$$

respectively, where $\theta_{c,r}=\operatorname{arctg}\sqrt{\dfrac{r^2-c}{r^2+c}}$.

We introduce the contours

$$\hat{\Im}_c=\hat{\Im}_c^+\cup\hat{\Im}_c^-,\hat{\Im}_c^{\pm}=\left\{z:\pm z=\sigma e^{-\frac{3\pi}{8}i},\sigma\in\left[2c\sqrt{1+\sqrt{2}},\infty\right),\right\}\cup$$

$$\cup\left\{z:\pm z=c(1+i\eta),\eta\in\left[-1-\sqrt{2};1+\sqrt{2}\right]\right\}\cup\left\{z:\pm z=\sigma e^{\frac{3\pi}{8}i},\sigma\in\left[2c\sqrt{1+\sqrt{2}},\infty\right)\right\}.$$

We denote a part of contours $\Im_c,\Im_c^{\pm},\hat{\Im}_c,\hat{\Im}_c^{\pm}$ enclosed inside the circle Ω_r, by $\Im_{c,r},\Im_{c,r}^{\pm},\hat{\Im}_{c,r},\hat{\Im}_{c,r}^{\pm}$ respectively. Al last, by Error! Objects cannot be created from editing field codes. for $r\ge 2c\sqrt{1+\frac{\sqrt{2}}{2}}$ we denote closed contours

$$\Gamma_{c,r}=\Omega_r(\theta_{c,r}+\pi,-\theta_{c,r}+2\pi)\cup\Im_{c,r}^+\cup\ \Omega_{c,r}(\theta_{c,r},-\theta_{c,r}+\pi)\cup\Im_{c,r}^-\ ,$$

$$\Gamma_{c,r}^+=\Im_{c,r}^+\cup\Omega_r(-\theta_{c,r},\theta_{c,r}),\ \hat{\Gamma}_{c,r}^+=\hat{\Im}_{c,r}^+\cup\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right),$$

$$\hat{\Gamma}_{c,r}=\Omega_r\left(-\frac{5\pi}{8},\frac{5\pi}{8}\right)\cup\hat{\Im}_{c,r}^+\cup\Omega_r\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)\cup\hat{\Im}_{c,r}^-.\quad (13)$$

Let $\{r_n\}$ be a sequence of such numbers that

$$0<r_1<r_2<\ldots<r_n<\ldots,\lim_{n\to\infty}r_n=\infty,$$

The circles Ω_{r_n} do not intersect the δ vicinity (δ is rather small, fixed) of the points $\mu_\nu\in S$. The number of points lying inside μ_ν ,$\hat{\Gamma}_{h,r_n}$ (h from (9)) is denoted by m_n. It is seen from (12) that for the functions $f(x)\in C^2[0,1], l_j f\big|_{x=0}=0$, $(j=2,3)$

$$\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}}\mu d\mu\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=f(x)+$$

$$+\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}}\mu^{-1}d\mu\int_0^1 G_1(x,\xi,\mu)\left[a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)\right]d\xi,$$

i.e.

$$f(x)=\sum_{\nu=1}^{m_n}\underset{\mu_\nu}{res}\,\mu\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi-$$

$$-\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}}\mu^{-1}d\mu\int_0^1 G_1(x,\xi,\mu)\left[a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)\right]d\xi.$$

Due to the estimation (10) and analyticity of the function $G_1(x,\xi,\mu)$ in the domain $\{\mu:\ \pm\operatorname{Re}\mu>h\}$ we have:

$$\lim_{n\to\infty}\int_{\hat{\Gamma}_{h,r_n}}\mu^{-1}d\mu\int_0^1 G_1(x,\xi,\mu)[a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)]d\xi=$$

$$\lim_{n\to\infty}\int_{\Omega_{r_n}}\mu^{-1}d\mu\int_0^1 G_1(x,\xi,\mu)[a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)]d\xi,$$

$$\left|\int_{\Omega_{r_n}}\mu^{-1}d\mu\int_0^1 G_1(x,\xi,\mu)[a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)]d\xi\right|\le$$

$$\le\int_{\Omega_{r_n}}\left|\frac{d\mu}{\mu}\right|\int_0^1|G_1(x,\xi,\mu)||a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)|\,d\xi\le$$

$$\le\int_0^{2\pi}\left|\frac{d\mu}{\mu}\right|\cdot\frac{c}{|\mu|}\cdot M=\int_0^{2\pi}\frac{K_0}{r_\nu}d\varphi\underset{\nu\to\infty}{\longrightarrow}0,$$

$$|a(\xi)f''(\xi)+b(\xi)f'(\xi)+c(\xi)f(\xi)|\le M,$$

thus,

$$f(x)=\lim_{n\to\infty}\sum_{\nu=1}^{m_n}\underset{\mu_\nu}{res}\,\mu\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=$$

$$=\sum_{\nu=1}^{\infty}\underset{\mu_\nu}{res}\,\mu\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi\qquad(14)$$

converges uniformly with respect to $x\in[0,1]$.

We prove the following theorem:

Theorem 1: Let $a(0)\alpha_0\beta_1+a(1)\beta_0\alpha_1\neq 0$, $\varphi(x)\in C^2[0,1]$, $l_j\varphi\big|_{x=0}=0$, (j = 2,3), the functions $a(x),b(x),c(x)$ be continuous in the interval $[0,1]$, $a(0)a(1)\neq 0$ and $a(x)>0$, $x\in[0,1]$. Then problem (1)-(4) can have at most one solution.

Proof: We introduce the operators

$$A_{\nu s}[f(x)]=\underset{\mu_\nu}{res}\,\mu^{2s+1}\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=f_{\nu s}(x), \qquad (15)$$

taking each function $f(x)\in C[0,1]$ to $f_{\nu s}(x)\in C^2[0,1]$, $l_j f_{\nu s}(x)\big|_{x=0}=0$, (j= 2,3) It is seen from (14) that if $f(x)\in C^2[0,1]$ and $l_j f\big|_{x=0}=0$, (j=2,3) then

$$\sum_{\nu=1}^{\infty} f_{\nu 0}(x)=f(x) \quad (16)$$

Note that if problem (1)-(4) has some solution $u(x,t)$, this function is the solution of problem (1), (2), (4) as well in the domain $\{(x,t):0<x<1,0<t\le\omega\}$. Since the operator of the problem (1)-(4) is hyper elliptic, using conditions 1)-3) from the definition of the solution, it is easy to see that the solution of problem (1), (2), (4) and, its derivatives u_t,u_{xx} for each $t\in(0,\omega]$ are continuous with respect to $x\in[0,1]$. Therefore, applying to (1), (2) the operators $A_{\nu s}$, we obtain

$$A_{\nu s}\left(\frac{\partial u}{\partial t}\right)=A_{\nu s}\left(a(x)\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial u}{\partial x}+c(x)u(x,t)\right)$$

$$\underset{\mu_\nu}{res}\,\mu^{2s+1}\int_0^1 G_1(x,\xi,\mu)\frac{\partial u(\xi,t)}{\partial t}d\xi=\underset{\mu_\nu}{res}\,\mu^{2s+1}\int_0^1 G_1(x,\xi,\mu)\times$$

$$\times\left[a(\xi)\frac{\partial^2 u}{\partial \xi^2}+b(\xi)\frac{\partial u}{\partial \xi}+c(\xi)u(\xi,t)\right]d\xi,$$

$$\frac{\partial}{\partial t}\underset{\mu_\nu}{res}\,\mu^{2s+1}\int_0^1 G_1(x,\xi,\mu)u(\xi,t)d\xi=\underset{\mu_\nu}{res}\,\mu^{2s+1}\int_0^1 G_1(x,\xi,\mu)\mu^2 u(\xi,t)d\xi,$$

$$\frac{\partial u_{\nu s}(x,t)}{\partial t}=u_{\nu s+1}(x,t),u_{\nu s}(x,0)=\varphi_{\nu s}(x). \quad (17)$$

The multiplicity of the pole µ_ν of the Green's function $G(x,\xi,\mu)$ will be denoted by χ_ν. Then it is clear that

$$A_{\nu_0}\left[\left(\mu^2-\mu_\nu^2\right)^{\chi_\nu}\right]u(x,t)=0, \qquad (18)$$

where we have:

$$A_{\nu_0}\left[\sum_{k=0}^{\chi_\nu}C_{\chi_\nu}^k\mu^{2k}\left(-\mu_\nu^2\right)^{\chi_\nu-k}u(x,t)\right]=0,$$

$$\sum_{k=0}^{\chi_\nu}C_{\chi_\nu}^k\left(-\mu_\nu^2\right)^{\chi_\nu-k}A_{\nu_0}\left[\mu^{2k}u(x,t)\right]=0,$$

$$\sum_{k=0}^{\chi_\nu}C_{\chi_\nu}^k\left(-\mu_\nu^2\right)^{\chi_\nu-k}u_\nu(x,t)=0,$$

i.e.

$$u_{\nu\chi_\nu}(x,t)=-\sum_{k=0}^{\chi_\nu-1}C_{\chi_\nu}^k\left(-\mu_\nu^2\right)^{\chi_\nu-k}, \qquad (19)$$

Where $C_{\chi_\nu}^k=\dfrac{\chi_\nu!}{k!(\chi_\nu-k)!}$.

Taking into account (19) in (17) we obtain that the set of function $u_{\nu s}(x,t)$ ($s=\overline{0,\chi_\nu-1}$) is a solution to the problem:

$$\frac{du_{\nu 0}(x,t)}{dt}=u_{\nu 1}(x,t) \qquad (20)$$

$$\frac{du_{\nu\chi_\nu-2}(x,t)}{dt}=u_{\nu\chi_\nu-1}(x,t)$$

$$\frac{du_{\nu\chi_\nu-1}(x,t)}{dt}=-\sum_{k=0}^{\chi_\nu-1}C_{\chi_\nu}^k\left(-\mu_\nu^2\right)^{\chi_\nu-k}u_{\nu k}(x,t)$$

$$u_{\nu 0}(x,0)=\varphi_{\nu 0}(x),\ldots\ldots,u_{\nu\chi_\nu-1}(x,0)=\varphi_{\nu\chi_\nu-1}(x), \qquad 21$$

the problem (20) and (21) has a unique solution and is represented by the formula

$$u_{\nu s}(x,t)=\underset{\mu_\nu}{res}\,\mu^{2s+1}e^{\mu^2 t}\int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi,$$

for the problem (20) s=0 and for the problem (21) s=0,1. Then by means of (16) we find

$$u(x,t)=\sum_{\nu=1}^{\infty}\underset{\mu_\nu}{res}\,\mu e^{\mu^2 t}\int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi \qquad (22)$$

Let problem (1)-(4) have two solutions $u_1(x,t)$ and $u_2(x,t)$ $(u_1(x,t)\neq u_2(x,t))$. Then their difference $w(x,t)=u_1(x,t)-u_2(x,t)$ will be the solution of the homogeneous problem (1)-(4) with $\varphi(x)=0$ and by the sense taken of the homogenous problem (1), (2), (4) in $\{(x,t):0\le x\le 1,0\le t\le\omega\}$.

Then by (22) $w(x,t)\equiv 0$ for $0\le x\le 1$, $0\le t\le\omega$ from conditions 2) it follows that $w(0,t)=w(1,t)=0$, for $t\ge 0$. In connection with these and condition 1) it is easy to see that the function

$$\upsilon(x,t)=\int_\omega^t w(x,\tau)d\tau$$

is the solution of the homogenous problem $\vartheta_t=a(x)\vartheta_{xx}+b(x)\vartheta_x+c(x)\vartheta$, $0<x<1,t\ge\omega$, $\upsilon(x,\omega)=0$ ($0\le x\le 1$), $w(0,t)=w(1,t)=0$, ($t>\omega$), continuous in ($0\le x\le 1,t\ge\omega$), then allowing for the maximum principle [30,31], we conclude that $\upsilon(x,t)\equiv 0$, ($0<x<1,t\ge\omega$), consequently, $\upsilon(x,t)\equiv 0$, ($0\le x\le 1,t\ge 0$).

Under the condition of theorem 1 and allowing for equality (12) we can reduce formula (22) to the form

$$u(x,t)=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}}\mu e^{\mu^2 t}d\mu\int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi=$$

$$=\varphi(x)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}}\mu^{-1}e^{\mu^2 t}d\mu\int_0^1 G_1(x,\xi,\mu)\times$$

$$\times\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi \qquad (23)$$

Keeping in mind the expression (13) of the contours $\hat{\Gamma}_{h,r_n}$ and $\hat{\Im}^-_{h,r_n}$ we know that $\hat{\Gamma}_{h,r_n}=\Omega_{r_n}\left(-\frac{5\pi}{8},-\frac{3\pi}{8}\right)\cup\hat{\Im}^+_{h,r_n}\cup\Omega_{r_n}\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)\cup\hat{\Im}^-_{h,r_n}$.

Now, estimating the function e^{μ^2t} on the arcs

$\Omega_{r_n}\left(-\frac{5\pi}{8}+j\pi,-\frac{3\pi}{8}+j\pi\right)$, ($j$=0,1), we have

$$\left|e^{\mu^2t}\right|=e^{t\operatorname{Re}\mu^2}=e^{t|\mu|^2\cos2\arg\mu}\le e^{-t|\mu|^2\frac{\sqrt2}{2}}=e^{-\frac{\sqrt2}{2}t|\mu|^2}.$$

Thus

$$\lim_{n\to\infty}\int\limits_{\Omega_{r_n}\left(-\frac{5\pi}{8}+j\pi,-\frac{3\pi}{8}+j\pi\right)}\mu^{-1}e^{\mu^2t}d\mu\int_0^1G_1(x,\xi,\mu)\times$$

$$\times\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi=0,(j=0,1).$$

Then from (23) we have

$$u(x,t)=\varphi(x)+\frac{1}{2\pi i}\int_{\hat{\Im}_h}\mu^{-1}e^{\mu^2t}d\mu\int_0^1G_1(x,\xi,\mu)\times$$

$$\times\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi$$

and using the property $G_1(x,\xi,-\mu)\equiv G_1(x,\xi,\mu)$ for solving the problem (1), (2), (4) we obtain the following formula

$$u(x,t)=\varphi(x)+\frac{1}{\pi i}\int_{\hat{\Im}^+_h}\mu^{-1}e^{\mu^2t}d\mu\int_0^1G_1(x,\xi,\mu)\times$$

$$\times\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi. \quad (24)$$

For $|\mu|>2h\sqrt{1+\frac{\sqrt2}{2}}$, i.e. on the distant parts of the contour $\hat{\Im}^+_h$ we have

$$\left|\frac{\partial^{k+m}}{\partial t^k\partial x^m}\mu^{-1}e^{\mu^2t}\int_0^1G_1(x,\xi,\mu)\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi\right|\le$$

$$\le C|\mu|^{2k+m-2}e^{-\frac{\sqrt2}{2}t|\mu|^2},(2k+m\le2). \quad (25)$$

Consequently, the operator ions $L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right),l_ju\big|_{x\to0}(j=2,3)$ for $0\le x\le1, 0\le t\le\omega$ can be taken under the integral sign (24) and then allowing for (11) we have

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u(x,t)=L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\left\{\varphi(x)+\frac{1}{\pi i}\int_{\hat{\Im}^+_h}\mu^{-1}e^{\mu^2t}d\mu\int_0^1G_1(x,\xi,\mu)\times\right.$$

$$\left.\times\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi\right\}=-\left[a(x)\varphi''(x)+b(x)\varphi'(x)+\right.$$

$$\left.+c(x)\varphi(x)\right]+\frac{a(x)\varphi''(x)+b(x)\varphi'(x)+c(x)\varphi(x)}{\pi i}\int_{\hat{\Im}^+_h}\mu^{-1}e^{\mu^2t}d\mu=$$

$$=-[a(x)\varphi''(x)+b(x)\varphi'(x)+c(x)\varphi(x)]+$$

$$+\frac{a(x)\varphi''(x)+b(x)\varphi'(x)+c(x)\varphi(x)}{2\pi i}\lim_{r\to\infty}\int_{\hat{\Gamma}_{h,r}}\mu^{-1}e^{\mu^2t}d\mu=0.$$

$$l_ju(x,t)\big|_{x\to0}=l_j\left[\varphi(x)+\frac{1}{\pi i}\int_{\hat{\Im}^+_h}\mu^{-1}e^{\mu^2t}d\mu\int_0^1G_1(x,\xi,\mu)\times\right.$$

$$\times[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)]]\big|_{x\to0}=l_j\varphi(x)\big|_{x\to0}+\frac{1}{\pi i}\int_{\hat{\Im}^+_h}\mu^{-1}e^{\mu^2t}d\mu\cdot$$

$$\cdot\int_0^1l_jG_1(x,\xi,\mu)\big|_{x\to0}\left[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)\right]d\xi=$$

$$=0+0=0,(j=2,3).$$

From formula (24) one can find the boundary values of the solution to problem (1), (2), (4).

$$u(s,t)=\varphi(s)+\frac{1}{\pi i}\int_{\hat{\Im}^+_h}\mu^{-1}e^{\mu^2t}d\mu\int_0^1G_1(x,\xi,\mu)\times$$

$$\times[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi)+c(\xi)\varphi(\xi)]d\xi\equiv\gamma_s(t),(s=0,1), \quad (26)$$

Note that integral (26) for $t\ge0$ and integrals from it formally differentiated with respect to t any number of times for $t\ge t_1>0$ ($t_1>0$ is arbitrary) uniformly converge.

Existence and presentation of the solution

Applying the integral operator $A[f]=\int_0^\infty e^{-\lambda^2t}f(t)dt$ (see. [6]) to the equation (1) and boundary condition (3), we obtain the following second spectral problem with a complex parameter λ:

$$L\left(\frac{}{dx},\lambda\right)z(x,\lambda)=-\varphi(x), \quad (27)$$

$$\begin{cases}e^{\lambda^2\omega}z(0,\lambda)+\delta_0z(1,\lambda)=A(\lambda)\\ z(0,\lambda)+\delta_1e^{\lambda^2\omega}z(1,\lambda)=B(\lambda),\end{cases} \quad (28)$$

where

$$L\left(\frac{d}{dx},\lambda^2\right)z(x,\lambda)=a(x)z''+b(x)z'+\left(c(x)-\lambda^2\right)z,$$

$$A(\lambda)=e^{\lambda^2\omega}\int_0^\omega e^{-\lambda^2t}u(0,t)dt,$$

$$B(\lambda)=\delta_1e^{\lambda^2\omega}\int_0^\omega e^{-\lambda^2t}u(1,t)dt.$$

Boundary conditions (28) can be reduced to the following form

$$z(0,\lambda)=p,z(1,\lambda)=q, \quad (29)$$

where

$$p = z_0(\lambda) = \left[\delta_1 e^{2\lambda^2\omega} - \delta_0\right]^{-1}\left(\gamma_1 A(\lambda) e^{\lambda^2\omega} - \gamma_0 B(\lambda)\right)$$

$$q = z_1(\lambda) = \left[\delta_1 e^{2\lambda^2\omega} - \delta_0\right]^{-1}\left(e^{\lambda^2\omega} B(\lambda) - A(\lambda)\right). \quad (30)$$

Let us consider the function

$$Q(x,\lambda,p,q) = \left[\exp\left(-\int_0^1\left(\frac{\lambda}{\sqrt{a(x)}} + \frac{b(x)}{2a(x)}\right)dx\right) - \exp\left(\int_0^1\left(\frac{\lambda}{\sqrt{a(x)}} - \frac{b(x)}{2a(x)}\right)dx\right]^{-1}\left[p\left(\exp\left(-\int_0^1\left(\frac{\lambda}{\sqrt{a(x)}} + \frac{b(x)}{2a(x)}\right)dx - q\right)\exp\left(\int_0^x\left(\frac{\lambda}{\sqrt{a(\xi)}} - \frac{b(\xi)}{2a(\xi)}\right)d\xi\right.\right.$$

$$.+(q- p(\exp\left(\int_0^1\left(\frac{\lambda}{\sqrt{a(x)}} + \frac{b(x)}{2a(x)}\right)dx\right)\left(\exp\left(-\int_0^x\left(\frac{\lambda}{\sqrt{a(\xi)}} - \frac{b(\xi)}{2a(\xi)}\right)d\xi\right]. \quad (31)$$

where p and q are determined by formula (30).

If , i.e. p and q are constants, then the function $Q(x,\lambda,p,q)$ except the points $\lambda_\nu = \left[\int_0^1 \frac{dx}{\sqrt{a(x)}}\right]^{-1} \nu\pi i + O\left(\frac{1}{\nu}\right),\ \nu \to \infty$ is everywhere analytic.

Obviously, at all the points of λ_ν where $Q(x,\lambda,p,q)$ the following identities exist and are valid $L\left(\frac{d}{dx},\lambda^2\right)Q(x,\lambda,p,q) = 0$ (32)

$$Q(0,\lambda,p,q) = p, Q(1,\lambda,p,q) = q.$$

It is known that for the spectral problem

$L\left(\frac{d}{dx},\lambda^2\right)z(x,\lambda) = 0, z(0,\lambda) = p, z(1,\lambda) = 0$, with a complex parameter λ, we have the Green's function $G_2(x,\xi,\lambda)$, analytical on λ everywhere, except for the points $\lambda_\nu = \left[\int_0^1 \frac{dx}{\sqrt{a(x)}}\right]^{-1} \nu\pi i + O\left(\frac{1}{\nu}\right)$, $\nu \to \infty$ which is its simple poles.

Note some known facts of the Green function $G_2(x,\xi,\lambda)$: there exists such $\delta > 0$ that on the λ plane outside the set $\bigcup_{\nu=1}^{\infty}\{\lambda : |\lambda - \lambda_\nu| < \delta\}$ the following estimation

$$\left|\frac{\partial^k G_2(x,\xi,\lambda)}{\partial x^k}\right| \le C \cdot |\lambda|^{k-1}, C > 0, k = 0,1,2, \quad (33)$$

is valid for all $x,\xi \in [0,1]$;

for $\lambda \ne \lambda_\nu$ ($\nu = 0,\pm 1,\dots$)

$$L\left(\frac{d}{dx},\lambda^2\right)G_2(x,\xi,\lambda)\varphi(\xi)d\xi = -\varphi(x),$$

$$G_2(0,\xi,\lambda) = G_2(1,\xi,\lambda) = 0. \quad (34)$$

Obviously, the solution of the second spectral problem is represented by the sum of two solutions:

$$z(x,\lambda) = -\int_0^1 G_2(x,\xi,\lambda)\varphi(\xi)d\xi + Q(x,\lambda,p,q), \quad (35)$$

We now fix the number $c_1 > \max\left(0, \ln\left|\frac{\delta_0}{\delta_1}\right|\right)$ and prove the following main theorems.

Theorem 2: Let $a(0)\alpha_0\beta_1 + a(1)\beta_0\alpha_1 \neq 0$, $\varphi(x) \in C^2[0,1]$, $l_j\varphi\big|_{x=0} = 0$ (j= 2,3), $a(x) > 0$, $x \in$ [0,1], $a(x) \in C[0,1]$, $b(x) \in C[0,1]$, $c(x) \in C[0,1]$, and $a(0)a(1) \neq 0$. Then the problem (1)-(4) has a solution and is represented by the following formula

$$u(x,t) = \varphi(x) + \frac{1}{\pi i}\int_{\tilde{\Im}_{c_1}^{+}} \lambda^{-1} e^{\lambda^2 t} \times$$

$$\times\left[\int_0^1 G_2(x,\xi,\lambda)\big(a(\xi)\varphi''(\xi) + b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)\big)d\xi - \right.$$

$$\left. -Q(x,\lambda,\varphi(0),\varphi(1)\right]d\lambda + \frac{1}{\pi i}\int_{\Im_{c_1}^{+}} \lambda e^{\lambda^2 t} Q(x,\lambda,p,q)\,d\lambda. \quad (36)$$

Proof: It is seen from formula (36) that the solution consist of three integrals and each of them is studied in the same way.

$$u_1(x,t) = \frac{1}{\pi i}\int_{\tilde{\Im}_{C_1}^{+}} \lambda^{-1} e^{\lambda^2 t} d\lambda \int_0^1 G_2(x,\xi,\lambda)[a(\xi)\varphi''(\xi) +$$

$$+b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)]d\xi, \quad (37)$$

$$u_2(x,t) = -\frac{1}{\pi i}\int_{\tilde{\Im}_{C_1}^{+}} \lambda^{-1} e^{\lambda^2 t}\, Q\big(x,\lambda,\varphi(0),\varphi(1)\big)d\lambda\,, \quad (38)$$

$$u_3(x,t) = \frac{1}{\pi i}\int_{\Im_{C_1}^{+}} \lambda e^{\lambda^2 t}\, z_1(x,\lambda,p,q)d\lambda\,. \quad (39)$$

On the distant parts of the contour $\tilde{\Im}_{C_1}^{+}$, i.e. Re $\lambda > C_1$

$$\left|e^{\lambda^2 t}\right| = e^{t|\lambda|^2 \cos 2\arg\lambda} = e^{t|\lambda|^2 \cos\left(\pm\frac{3\pi}{4}\right)} = e^{-\frac{\sqrt{2}}{2}t|\lambda|^2} \quad (40)$$

And by means of the estimation (33) it is clear that

$$u_1(x,t) \in C^{2,1}(0 \le x \le 1, t > 0) \cap C(0 \le x \le 1, t \ge 0) \quad (41)$$

And this enables us that the operators $\frac{\partial}{\partial t}, \frac{\partial^2}{\partial x^2}, x \to 0, x \to 1, t \to 0$ can be taken under the integral sign (37).

We have

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)u_1(x,t) = \frac{1}{\pi i}\int_{\tilde{\Im}_{C_1}^{+}} \lambda^{-1} e^{\lambda^2 t}\, d\lambda\; L\left(\frac{d}{dx}, \lambda^2\right) \times$$

$$\times \int_0^1 G_2(x,\xi,\lambda)[a(\xi)\varphi''(\xi) + b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)]d\xi =$$

$$= -\frac{a(x)\varphi''(x) + b(\xi x)\varphi'(x) + c(x)\varphi(x)}{\pi i}\int_{\tilde{\Im}_{C_1}^{+}} \lambda^{-1} e^{\lambda^2 t}\, d\lambda =$$

$$= -\frac{a(x)\varphi''(x) + b(\xi x)\varphi'(x) + c(x)\varphi(x)}{2\pi i}\left[\int_{\Im_{C_1}^{+}} \lambda^{-1} e^{\lambda^2 t}\, d\lambda + \int_{\Im_{C_1}^{-}} \lambda^{-1} e^{\lambda^2 t}\, d\lambda\right] =$$

$$= -\frac{a(x)\varphi''(x) + b(\xi x)\varphi'(x) + c(x)\varphi(x)}{2\pi i}\lim_{r\to\infty}\left[\int_{\Im_{C_{1,r}}^{-}} \lambda^{-1} e^{\lambda^2 t}\, d\lambda + \right.$$

$$\left. + \int_{\Omega_r\left(-\frac{5\pi}{8},-\frac{3\pi}{8}\right)} \lambda^{-1} e^{\lambda^2 t}\, d\lambda + \int_{\Im_{C_{2,r}}^{+}} \lambda^{-1} e^{\lambda^2 t}\, d\lambda + \int_{\Omega_r\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)} \lambda^{-1} e^{\lambda^2 t}\, d\lambda \right] =$$

$$= -\left(a(x)\varphi''(x) + b(x)\varphi'(x) + c(x)\varphi(x)\right), (42)$$

for $t > 0$. By means of (40) the integrals on the arcs $\Omega_r\left(-\frac{5\pi}{8}, -\frac{3\pi}{8}\right)$, $\Omega_r\left(\frac{3\pi}{8}, \frac{5\pi}{8}\right)$ tend to zero as $r\to\infty$.

The function $G_2(x,\xi,\lambda)$ is analytic in the domain Re $\lambda>0$, and using the estimation (33) we find

$$u_1(x,0) = \frac{1}{\pi i}\int_{\Im_{C_1}^{+}} \lambda^{-1} d\lambda \int_0^1 G_2(x,\xi,\lambda)[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)]d\xi$$

$$= \frac{1}{\pi i}\left[\int_{\Im_{C_1}^{+}} \lambda^{-1} d\lambda \int_0^1 G_2(x,\xi,\lambda)[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)]d\xi \right. +$$

$$\left. + \int_{\Omega_r\left(-\frac{5\pi}{8},-\frac{3\pi}{8}\right)} \lambda^{-1}\, d\lambda \int_0^1 G_2(x,\xi,\lambda)[a(\xi)\varphi''(\xi)+b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)]d\xi \right] = 0, \quad (43)$$

and for $t>0$ by (33), (40) and equalities (34) we have

$$u_1(s,t) = \frac{1}{\pi i}\int_{\Im_{C_1}^{+}} \lambda^{-1} e^{\lambda^2 t} d\lambda \int_0^1 G_2(x,\xi,\lambda)[a(\xi)\varphi''(\xi) +$$

$$+b(\xi)\varphi'(\xi) + c(\xi)\varphi(\xi)]d\xi = 0, (s = 0, 1). (44)$$

We now study the second integrals $u_2(x,t)$. It is seen from formula (31) that the function $Q(x,\lambda,\varphi(0),\varphi(1))$ in the domain Re $\lambda>c_1$ is analytic and the following estimations are valid for it

$$|Q(x,\lambda,\varphi(0),\varphi(1))| \le c_1 e^{-|\lambda|\int_0^x \frac{d\xi}{\sqrt{a(\xi)}}\cos\frac{3\pi}{8}} + c_2 e^{-|\lambda|\left(1-\int_0^x \frac{d\xi}{\sqrt{a(\xi)}}\right)\cos\frac{3\pi}{8}} + \frac{c_3}{|\lambda|}, \quad (45)$$

on the distant parts (Re $\lambda>c_1$) of the contour $\Im_{C_1}^{+}$ and on the arcs $\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right)$, $r > 2c_1\sqrt{1+\sqrt{2}}$ and the estimation

$$\left|\frac{\partial^k Q(x,\lambda,\varphi(0),\varphi(1))}{\partial x^k}\right| \le c_4|\lambda|^k e^{-|\lambda|\int_0^x \frac{d\xi}{\sqrt{a(\xi)}}\cos\frac{3\pi}{8}} +$$

$$+c_5|\lambda|^k e^{-|\lambda|\left(1-\int_0^x \frac{d\xi}{\sqrt{a(\xi)}}\right)\cos\frac{3\pi}{8}} + \frac{c_6}{|\lambda|^{k+1}}, \qquad (k=0,1,2) \qquad (46)$$

for all x∈[0,1].
If follows from (40) and (46) that

$$u_2(x,t) \in C^{2,1}(0 \le x \le 1, t > 0)$$

and in (38) for t>0 the operations $L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)$, $x \to 0, x \to 1$, can be taken under integral sign.

Then allowing for (32), we obtain

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u_2(x,t) = 0,$$

$$u_2(s,t) = -\frac{\varphi(s)}{\pi i}\int_{\hat{\Im}^+_{C_1}} \lambda^{-1}e^{\lambda^2 t}d\lambda = -\frac{\varphi(s)}{\pi i}\lim_{r\to\infty}\int_{\hat{\Im}^+_{C_{1,r}}} \lambda^{-1}e^{\lambda^2 t}\,d\lambda = -\varphi(s), (s=0,1),$$

As can be seen from (45), for *x*, belonging to any segment of $[x_1,x_2] \in (0,1)$, the integral (38) converges uniformly with respect to $t \ge 0$. Then $u_2(x,t) \in C(0 < x < 1, t \ge 0)$ and for $x \in [x_1,x_2]$

$$u_2(x,0) = \frac{1}{\pi i}\int_{\hat{\Im}^+_{C_1}} \lambda^{-1}\,Q(x,\lambda,\varphi(0),\varphi(1))d\lambda = \frac{1}{\pi i}\times$$

$$\times\lim_{r\to\infty}\left[\int_{\hat{\Im}^+_{C_{1,r}}} \lambda^{-1}Q(x,\lambda,\varphi(0),\varphi(1))\,d\lambda +\right.$$

$$\left.+\int_{\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right)} \lambda^{-1}\,Q(x,\lambda,\varphi(0),\varphi(1))d\lambda\right] = 0 \quad (47)$$

where the function $Q(x,\lambda,\varphi(0),\varphi(1))$ is analytic inside the closed contour $\hat{\Gamma}^+_{C_1,r}$.

Also, $u_3\,(x,t)$ is studied in the same way. Combining theorems 1, 2, we arrive at the following final statement

Theorem 3: Let $a(0)\alpha_0\beta_1 + a(1)\beta_0\alpha_1 \ne 0$. , $\varphi(x) \in C^2[0,1]$ and $l_j\varphi|_{x=0} = 0$ $(j=2,3)$. Then problem (1)-(4) has a unique solution represented by formula (36).

Conflict of Interest
The authors declare that they have no conflict of interest relevant to the content of this manuscript.

References

1. Bitsadze A, Samarskii A (1969) On some simplest generalizations of linear elliptic problems. Dokl Akad Nauk SSSR 185: 739-740.
2. Anderson D (1998) Multiple positive solutions for a three-point boundary value problem. Math Comput Model 27: 49-57.
3. Ma R (2001) Positive solutions of a nonlinear m-point boundary value problem. Comput Math Appl 42: 755-765.
4. Domínguez-Pérez M, Rodríguez-López R (2012) Multipoint boundary value problems of Neumann type for functional differential equations. Nonlinear Anal Real World Appl 13: 1662-1675.
5. Gurevich P (2008) Smoothness of generalized solutions for higher-order elliptic equations with nonlocal boundary conditions. J Differ Equ 245: 1323-1355.
6. Pečiulyté S, Štikonas A (2006) Sturm-Liouville Problem for Stationary Differential Operator with Nonlocal Two-Point Boundary Conditions. Nonlin Anal Model Control 11: 47-78.
7. Ma R (2007) A survey on nonlocal boundary value problems. Applied Mathematics E-Notes 7: 257-279.
8. Calvert BD (2010) One-dimensional nonlinear Laplacians

under a 3-point boundary condition. Acta Math Sin Engl Ser 26: 1641-1652.
9. Wang Y, Ge W (2007) Multiple positive solutions for multipoint boundary value problems with one-dimensional p-Laplacian. J Math Anal Appl 327: 1381-1395.
10. Calvert BD (2010) Stability of equilibrium solution to inhomogeneous heat equation under a 3-point boundary condition. Appl Anal 89: 409-414.
11. Tsankov Y (2010) Explicit solutions of nonlocal boundary value problems, containing Bitsadze-Samarskii constraints. Fract Calc Appl Anal 13: 435-446.
12. Alikhanov A (2008) Nonlocal boundary value problems in differential and difference settings. Differ Equ 44: 952-959.
13. Alikhanov A (2010) On the Stability and Convergence of Nonlocal Difference Schemes. Differ Equ 46: 949-961.
14. Marizán Slodiçka (2021) On a semilinear parabolic problem with four-point boundary conditions Mathematics 9: 468.
15. Yeliseev A, Ratnikova T, Shaposhnikova D (2021) Regularized asymptotics of the solution of the singularly perturbed first boundary value problem on the semiaxis for a parabolic equation with a rational “Simple” turning point. Mathematics 9: 405.
16. Aitzhanov SE, Berdyshev AS, Bekenayeva KS (2021) Solvability issues of a pseudo-parabolic fractional order equation with a nonlinear boundary condition. Fractal Fract 5: 134.
17. Ozbilge E, Kanca F, Özbilge E (2022) Inverse problem for a time fractional parabolic equation with nonlocal boundary conditions. Mathematics 10: 1479.
18. Mammadov YA, Ahmadov HI (2020) Mixed problem for the heat equation with advance in boundary conditions. Russ Math 64: 25-42.
19. Ahmadov HI (2023) The solution of a mixed problem for a parabolic type equation with general form coefficients under unconventional boundary conditions. Azerbaijan Journal of Mathematics 13:172-187.
20. Ahmadov HI (2023) On a mixed problem for a parabolic type equation with general form constant coefficients under lnhomogeneous boundary conditions. American Journal of Appl Math 11: 32-39.
21. Rasulov ML (2014) Methods of Contour Integration. North-Holland. https://books.google.co.in/books/about/Methods_of_Contour_Integration.html?id=eEOoBQAAQBAJ&redir_esc=y
22. Rasulov ML(1975) Applications of the contour integral method. Nauka Moscow https://trans.imm.az/volumes/38-4/3804-08.pdf.
23. Utkina EA (2015) Characteristic boundary value problem for a fourth-order equation with a pseudo parabolic operator and with shifted arguments of the unknown function. Differential Equation 51: 426-429.
24. Wu I(1996) Theory and Applications of Partial Functional Differential Equations
25. https://books.google.co.in/books/about/Theory_and_Applications_of_Partial_Funct.html?id=qWL9zc0b7_0C&redir_esc=y
26. Kamunin LI (1964) A boundary value problem in the theory of heat conduction with a nonclassical boundary condition. USSR Computational Mathematics and Mathematical Physics 4: 33-59.
27. Teo KL (1977) Second order linear partical differential of parabolic type with delayed arguments. Nauka Math 10: 119-130.
28. Podgornov VV (1967) The first boundary value problem for quasilinear parabolic equations with retarded argument. Differential Equations 3: 1334-1341.
29. Ionkin NI (1977) Solution of a boundary-value problem in heat conduction with a non-classical boundary condition. Differential Equations 13: 204-211.
30. Naimark MA (1968) Linear Differential Operators http://opacrri.rri.res.in/cgi-bin/koha/opac detail.pl?biblionumber=13572&query_desc=su%3A%22Differential%20equations%22 .
31. Yu A Mamedov, Ahmadov HI (2004) Almost Regularity Conditions of Spectral Problems for a Second Order Equation. Commun Theor Phys 41: 649.
32. Vladimirov VS (2021) Equations of mathematical physics. M Mir Publishers https://mirtitles.org/2021/06/18/equations-of-mathematical-physics-vladimirov/.