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Reconstruction of a Band Limited Signal Sampled Slower than the Nyquist Rate Using AM Modulated Version of the Signal

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ABSTRACT

In this paper, a modified sampling rate, which is smaller than Nyquist rate, is determined for bandlimited signals using Amplitude Modulation (AM). The proposed method is novel and valid for impulse sampling in continuous time.

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Introduction

This paper presents a new method for data reconstruction of a signal from its samples which are taken slower than the Nyquist rate.

The reconstruction of a signal from the samples can be achieved if the samples of the signal are taken at a sampling frequency f_s which is larger than the bandwidth of the sampled signal. This rate is known as Shannon's sampling theorem in the communication community since it was introduced and applied to communication area by Shannon in 1949 [1].

The faster than Nyquist (FTN) is an approach, which samples a low-frequency signal $f_l(t)$ instead of a high frequency band limited signal $f_h(t)$, which has the spectrum of $f_l(t)$ at high frequency. Then $f_h(t)$ can be obtained from the samples of $f_l(t)$ via a frequency shift [2]. In FTN the signal is sampled at a lower sampling frequency than the one required by $f_h(t)$ based on the similarity of the spectrums of $f_l(t)$ and $f_h(t)$. For FTN, the high frequency signal should resemble the low frequency signal in terms of the spectrum. However, the proposed approach here can be applicable to all band-limited signals.

Here, the proposed method has the potential to increase the sampling period $T = \frac{1}{f_S}$ up to twice the value proposed by the Nyquist theory, where f_S is the sampling frequency. In this paper, the following issues are presented: i) the reconstruction of the under sampled signal; ii) the conjecture that the samples of a signal can be obtained by adding enough AM signals at the center frequencies, $kw_s=2\pi kf_s$, which are the integral multiples of the sampling frequency.

Sampling Theory and AM

Sampling Theory

Let a band limited signal, $g(t)$, be given (Figure 1). The signal $g(t)$ has a bandwidth, $BW = 2f_{max}$, where f_{max} is the maximum frequency component of $g(t)$.

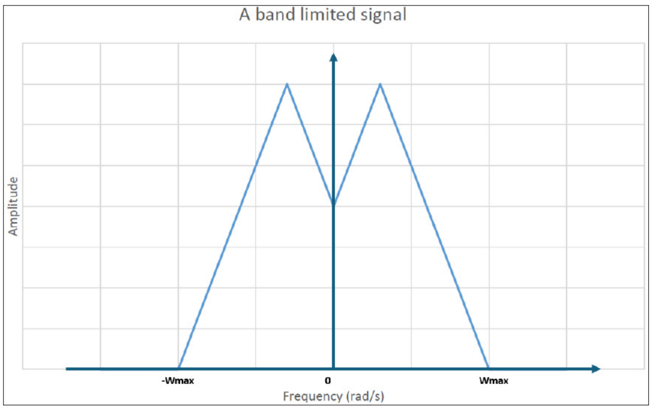

Figure 1: Spectrum of a Band Limited Signal

The signal $g(t)$ is desired to be stored in a digital system through the samples taken at every T second, such that, the original signal, $g(t)$, can be reconstructed from the samples without a loss of information.

The sampling process is formulated by use of the impulse train called Dirac comb (Figure 4a) of period T [3],

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT). \quad (1)$$

Note that, the limits of summation will be the set of integers from -∞ to +∞ unless otherwise stated.

The sampled signal can be represented as follows [4]

$$g_T(t) = g(t)\delta_T(t) = \sum g(t)\delta(t-nT). \qquad (2)$$

Using the properties of impulse functions we have

$$g^*(t) = g_T(t) = \sum g(nT)\delta(t-nT). \qquad (3)$$

That is, we get a weighted impulse train by the weights equal to $g(nT)$ (Figure 4b).

For a causal system, the Laplace transform of the Dirac Comb given in (1) is

$$\mathcal{L}(\delta_T(t)) = \int_0^\infty \delta_T(t)e^{-st}dt = \sum_0^\infty e^{-nTs}. \qquad (4)$$

Note that, Laplace transform may be defined as two-sided and one-sided versions. For causal systems, both definitions are equivalent [5].

Similarly, the Laplace transform of $G^*(s)$ is

$$G^*(s) = \sum_0^\infty g(nT)e^{-nTs} = \sum_0^\infty g(nT)\,(e^{sT})^{-n}. \qquad (5)$$

In Figure 2, spectrum of $|G^*(jw)|$ is seen for a sampling frequency higher than Nyquist frequency, while a spectrum with aliasing is given in Figure 5. $G^*(s)$, the impulse sampled function, has some useful properties, listed below:

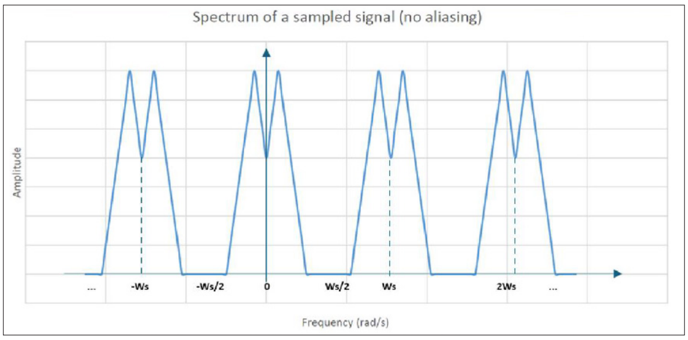

Figure 2: Spectrum $|G^*(jw)|$ of a sampled signal (no aliasing), that is, $w_s > 2w_{max}$

- $G^*(s)$ is periodic in s with the period $j\frac{2\pi}{T} = jw_s$, that is
- (Figure 2)

$$G^*(s) = G^*\left(s + j\frac{2\pi}{T}k\right);\ \forall k \in \mathbb{Z}. \qquad (6)$$

- If $G(s)$ has a pole at $s=s_1$, then $G^*(s)$ has poles at

  $s = s_1 + jkw_s;\ \forall k \in \mathbb{Z}.$

  Note that this is not true for the zeros of $G(s)$ in general.
- $G^*(s)$ can be expressed as follows:

$$G^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + jkw_s). \qquad (7)$$

- In terms of Fourier Transformation, the frequency response $G^*(jw)$ is the repeated version of $\frac{G(j\bar{w})}{T}$ shifted by the integral multiples of sampling frequency, w_S along the jw- axis.
  If the sampling frequency satisfies the Nyquist rate, then the signal can be recovered from the samples.

- Since the respective conditions $w_s > 2w_{max}$ and $w_c > w_{max}$ (see Figure 7) for sampling and AM are similar, one can conclude that $G^*(jw)$ includes the AM modulated versions of $g(t)$ infinitely many times with carrier frequencies of kjw_s, for all integer k. Note that, the AM condition should be same to satisfy the sampling conditions to extract AM signal from the spectrum of a sampled signal using a proper bandpass filter (Figure 2).
- Therefore, the AM modulation of $g(t)$ with carrier frequency, kw_s for a given k, can be extracted from $G^*(s)$ using an appropriate bandpass filter with $BW=w_s$ centered at kw_s.

The recovery of the signal is not possible due to the aliasing phenomenon if the sampling frequency is less than twice the maximum frequency of the signal according to the Nyquist theorem [6].

Amplitude Modulation

The basic AM signal, $m_a(t)$, can be expressed as the multiplication of the carrier signal with the message signal as follows:

$$m_a(t) = m(t)cos(w_c t) \qquad (8)$$

where $m(t)$ is the band-limited message signal with a maximum frequency of w_{max}, $\cos(w_c t)$ is the carrier function with the carrier frequency, $w_c > 2w_{max}$.

Since the AM is based on frequency contents, the AM is analyzed using Fourier Transform. The Fourier Transform of $m_a(t)$ is given by (Figure 3)

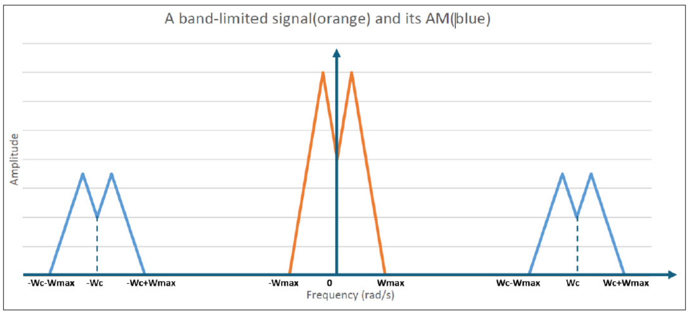

Figure 3: A band limited signal (center) and its AM signal at w_C (left and right)

$$M_a(jw) = \mathcal{F}\{cos(w_c t)m(t)\} = \frac{1}{2}\mathcal{F}\{[e^{-jw_c t} + e^{jw_c t}]m(t)\}$$

$$M_a(jw) = \frac{1}{2}\big(M(jw + jw_c) + M(jw - jw_c)\big). \qquad (9)$$

Reconstruction with AM

In this section, a novel method is offered to recover an under-sampled signal.

Let us assume that the frequency response of $g(t)$ is symmetric and $G^*(s)$ is the Laplace transform of the impulse sampled version with $w_{max} < w_S < 2w_{max}$. Due to non-Nyquist choice of sampling frequency, $G^*(jw)$ exhibits aliasing and $g(t)$ cannot be reconstructed from the samples at hand (see Figure 5).

Let us assume that the $G(s+jw_s)$ and $G(s-jw_s)$ are available. Let $G^{**}(s)$ be defined as

$$G^{**}(s) = G^*(s) - \frac{G(s+jw_S)+G(s-jw_S)}{T}. \qquad (10)$$

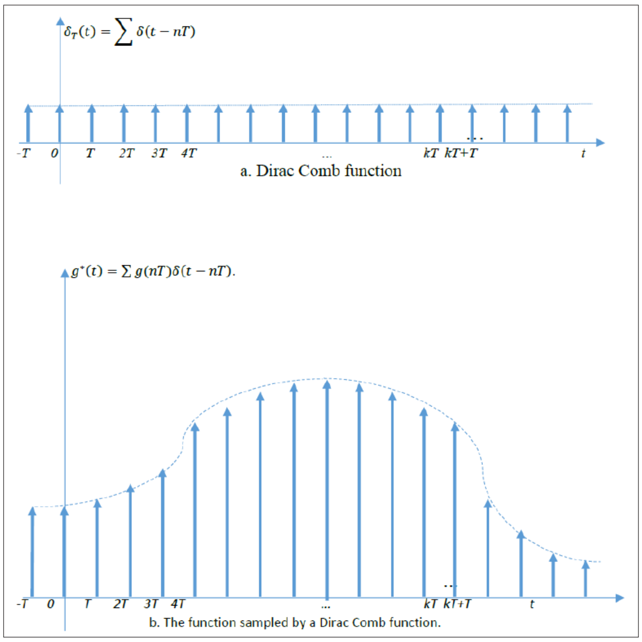

Figure 4: (a) The Dirac (impulse train) comb function; (b) the samples $g^*(t)$ of a function with sampling frequency w_s=1/T

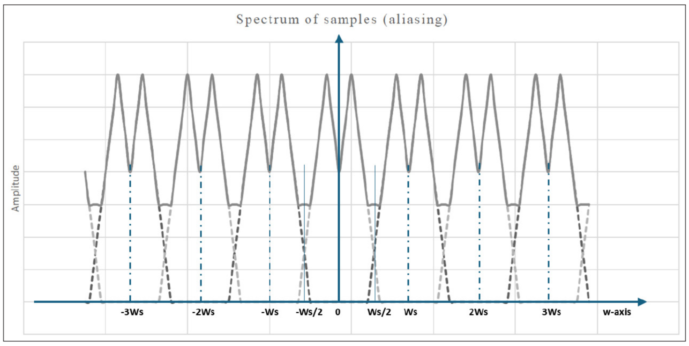

Figure 5: The spectrum of an under-sampled signal ($|G^*(jw)|$ with $w_s<2w_{max}$ (aliasing) Note: Only $\pm w_s/2$ is shown!

Then $G^{**}(jw)$ has a non-aliased version of $G(jw)$ around w=0 because the parts (Figure 6), which cause aliasing, are removed. Then using an appropriate low pass filter, we can recover the original signal from the samples even though $g(t)$ was sampled with non-Nyquist sampling frequency.

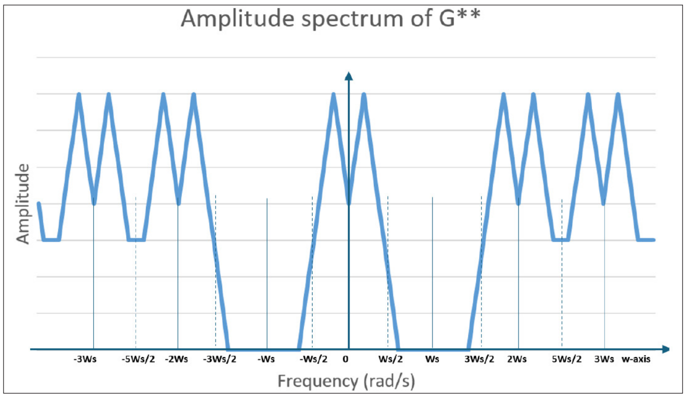

Figure 6: The spectrum of $|G^{**}(jw)|$. Aliasing is removed

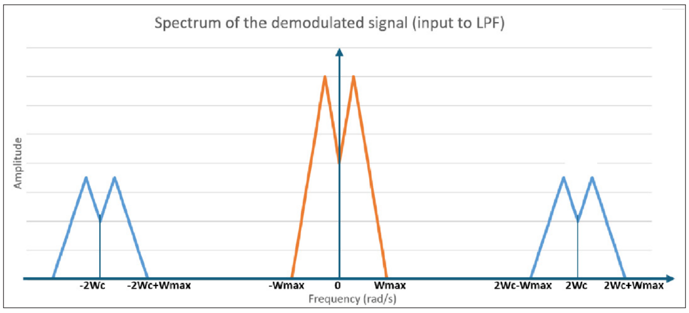

Figure 7: The spectrum of a demodulated AM signal before LPF. Aliasing occurs when $w_c<w_{max}$.

Note that, considering the Fourier transforms, we can write

$$G(s + jw_s) + G(s - jw_s) = 2G_a(s). \qquad (11)$$

where $g_a(t)$ is the AM modulated $g(t)$. This means that, a weighted version of AM signal of $g(t)$ should be removed from $G^*(s)$. On the other hand, shifting $G(s)$ properly provides the same result.

Now, according to the above approach, it seems that we can remove all parts that results aliasing for smaller sampling frequencies. Thus, we can recover the signal from very few samples. The following remarks should be noted:

- It is possible to think as follows: If we have an AM modulated version of $g(t)$ at hand, we do not need to have samples of the signal for recovery. The AM wave can be used to recover it. Indeed, this is true if the AM carrier frequency is larger than the Nyquist rate. If the carrier has a frequency less than the Nyquist frequency, the signal cannot be obtained from AM without distortion. Therefore, for an under-sampled signal, the above procedure can be employed for reconstruction.
- On the other hand, if we think that enough AM versions are at hand, then the signal can be recovered even from a single sample. This clearly makes no sense. At least there should be a limit for recovery. The best-known limit is the Nyquist rate, which requires the sampling frequency to be twice the maximum frequency of $G(jw)$. Note that the AM condition and the Nyquist rate are same for a full recovery.
- However, the above approach inspires that there should be a method to recover the signal from its samples, which are taken slower than the Nyquist rate. The method should make possible to remove the aliasing, thus, allowing the recovery without a need to have an AM version at hand.

The approach here suggests that the reconstruction of a signal from its samples may be achieved for a non-Nyquist sampling period. The study continues to devise such a method. The approach presented is the preliminary results for recovery of an under-sampled signal. The G** method is a proof that signal recovery can be achieved with a sampling frequency less than Nyquist rate.

Obtaining Samples Using AM

In this section, it is conjectured that the sampling of a signal can be achieved by addition of sufficiently many AM signals based on the above approach.

Let us define the $am_k(t)$ function for a band limited $m(t)$ with $M(jw)$=0 for all $w>w_{max}$ as follows:

$$am_k(t) = m(t)\cos(kw_s t) \qquad (12)$$

where k is an integer, $w_s=2\pi f_s=2\pi/T>2\pi f_{max}=2w_{max}$ is the sampling frequency determined by the sampling period T.

From (9), the Fourier Transform of $am_k(t)$ is

$$AM_k(jw) = \frac{1}{2}\left(M(jw + jkw_s) + M(jw - jkw_s)\right) \quad (13)$$

Indeed, $am_k(t)$ is the amplitude modulation of $m(t)$ performed at the carrier frequency of $w_c=kw_s$.

From (8) and (13), (7) can be written as

$$G^*(s) = \frac{1}{T}\{M(jw) + 2\sum_{k=1}^{\infty} AM_k(jw)\} \quad (14)$$

From (14), one can predict that the samples of m(t), which are taken at every T second, can be obtained from $am_k(t)$.

Based on (14), the following conjecture can be stated as follows:

Conjecture 1

Let $m(t)$ be a band limited signal, whose spectrum is zero for all $w=2\pi f>2\pi f_{max}=w_{max}$. Then the samples of $m(t)$, which are taken at every T second that satisfies the Nyquist condition (that is, $f_s>2f_{max}$), can be obtained by adding sufficiently many AM modulations, $am_k(t)$ of $m(t)$ without using a sampling device.

The proof of the above claim has not been constructed yet. Also, the expression "many" should be figured based on the parameters and properties of the sampled signal. The study is continuing.

Conclusion

A signal recovery method is proposed for under sampled signals using AM signal of the sampled signal. The approach has a very limited application since it is valid in continuous time where the samples are available as the weights of shifted impulses of a Dirac comb. The proposed approach can remove the aliasing in continuous time. However, the aliasing cannot be avoided when the samples are considered as number sequences in discrete time due to the cyclic nature of Fourier transforms for discrete time sequences. The signal spectrum on the segment $\pm 0.5jw_s$ is placed infinitely many times onto the unit circle in z-domain under sampling. The strips of jw- axis are carried on the unit circle in discrete time setting while the weighted spectrum is repeated along jw-axis at every segments of the form $kjw_s \pm 0.5jw_s$ for all integers. In case of aliasing, the signal bandwidth, BW = $2w_{max}$ is larger than the sampling frequency, w_s. Although the aliasing can be avoided in continuous time using G^{**} defined in this study, The aliasing in discrete setting cannot be avoided using the same method due to the cyclic nature of the transforms for discrete time.

It is also conjectured that the samples of the signal (taken according to the Nyquist rate) can be obtained using sufficiently many AM signals at integer multiples of the sampling frequency mathematically without a need to use sampling devices. This result should be proved.

The search for a transformation or a method that prevents aliasing is continuing. As stated earlier, the study to achieve such result is going on based on the result obtained in continuous time setting. Similarly the study for the conjecture is continuing.

References

1. Shannon CE (1998) Communication in the Presence of Noise. Proceedings of the IEEE 86: 447-457.
2. Anderson JB, Rusek F, Owall V (2013) Faster-Than-Nyquist Signaling. In Proceedings of the IEEE 101: 1817-1830.
3. Fischer Jens V, Rudolf L Stens (2019) On Inverses of the Dirac Comb. MDPI Mathematics 7: 1196.
4. Philips CL, Nagle HT (1994) Digital Control System Analysis and Design 3rd Ed. Prentice Hall https://ece.ncsu.edu/book/digital-control-system-analysis-and-design-3rd-edition/.
5. Oppenheim Alan V, Willsky Alan S, Hamid Nawab S (1996) Signals and Systems 2nd Edition. Pearson https://www.pearson.com/en-us/subject-catalog/p/signals-and-systems/P200000003155/9780138147570.
6. Haykin S, Moher M (2009) Communication Systems, 5th Ed. John Wiley and Sons https://www.wiley.com/en-au/munication+Systems%2C+5th+Edition-p-9780471697909.