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Second Subgroup of General Linear Group in Dimention 2

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ABSTRACT

In many practical situations we know that a primitive group contains a given permutation and we want to know which group it can be; in some other practical situations we know the group and would like to know if it contains a permutation of some given type. For example, a group G ≤ Sn is said to be non-synchronizing if it is contained in the automorphism group of a non-trivial primitive graph with complete core, that is, with clique number equal to chromatic number. When trying to check if some group is synchronizing, typically, we have only partial information about the graph but enough to say that it has an automorphism of some type, and the goal would be to have in hand a classification of the primitive groups containing permutations of that type. As an illustration of this, the key ingredient in some of the results in [2] was the observation that the primitive graph understudy has a 2-cycle automorphism and hence the automorphism group of the graph is the symmetric group. For many more examples of the importance of knowing the groups that contain permutations of a given type. This type of investigation is certainly very natural since it appears on the eve of group theory, with Jordan, Burnside, Marggraff, but the difficulty of the problem is well illustrated by the very slow progress throughout the twentieth century. Given the new tools available (chiefly the classification of finite simple groups), the topic seems to have new momentum. Let Sn denote the symmetric group on n points; a permutation $g \in$ Sn is said to be imprimitive if there exists an imprimitive group contain in gg. An imprimitive group is said to beminimally imprimitiveifit contains no transitive proper subgroup. An imprimitive group G≤Sn is said to be maximally imprimitive if for all g Sn\G, the group 〈g, G〉 is primitive. The next result, whose proof isstraightforward, provides some alternative characterizations of imprimitive permutations. Now in this paper we discuse Presentation for imprimitive second Subgroup of general linear group in dimention 2 over the field of pk-elements .

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Elementary Definitions and Theorems

1.1. Definition: The Burnside poset *P*, of a group *G* is a poset with the following properties.

(i) Each element of *P* represents a conjugacy class of subgroups of *G* and each conjugacy class is represented exactly once in *P*.
(ii) If *C* and *D* are elements of *P*, then we write $C \leq D$ if and only if at least one group in the conjugacy class represented by *C* is a subgroup of at least one group in the conjugacy class represented by *D*.

1.2. Definition: We draw the Burnside inclusion diagram of the Burnside poset *P* of a group *G* according to the following rules .
(i) We represent elements of *P* by small black discs.
(ii) If an element of *P* represents a conjugacy class cantaining a single group (which is therefore normal in *G*) , we draw a circle around the disc representing that element.
(iii) We lable each disc with either the isomorphism type or the order of the groups in the conjugacy class it represents.
(iv) We represent the relation $C \leq D$ by placing the disc representing *C* lower on the page then the disc reperesenting *D* and by drowing a line between those two discs. Howerer , we suppress inclusion that are implied by the reflexiveness and transitivity of $\leq$.

1.3. Definition: Let $m \geq 2$ and $n \geq 3$ we will denote by I_n^{2m} (the I standing for' inversion') any group isomorphic to the following soluble group of order

$$2mn{:}< a,b \mid a^{2m}=1 \ ,b^{a} =b^{-1},b^{n}=1> .$$

1.4. Definition: We will denote by *SL*(2,3) any group isomorphic to the following soluble group of order 24 :

$$<a,b,c \mid a^{3}-1 ,$$
$$b^{a}=c \ , \ b^{2}=c^{2} ,$$
$$c^{a}=bc \ , \ c^{b}=c^{3}, \ c^{4}=1> .$$

1.5. Definition: We will denote by *GL*(2,3) any group isomorphic to the following soluble group of order 48 :

$$<a,b,c,d \mid a^{2}=1 ,$$
$$b^{a}=b^{2},b^{3}=1,$$
$$c^{a}=d^{3},c^{b}=d,c^{2}=d^{2}$$
$$d^{a}=cd^{2},d^{b}=cd,d^{c}=d^{3},d^{4}=1>$$

The Burnside inclusion diagram of $GL(2,3)$ is pictured in Figure 1. This can be checked via CAYLEY, or in the tables of Neubuser (1967) where $GL(2,3)$ has the number 48.49.

1.6. Definition: We will denote by BO (**for binary octahedral**) any group isomorphic to the following soluble group of order 48 :

$$
\begin{aligned}
<a,b,c,d \mid\ & a^2=d^2, \\
& b^a=b^2 \quad ,b^3=1 \quad , \\
& c^a=d^3 \quad ,c^b=d \quad ,c^2=d^2 , \\
& d^a=cd^2 \quad ,d^b=cd \quad , d^c=d^3 ,\ d^4=1>.
\end{aligned}
$$

The Burnside inclusion diagram of BO is pictured in Figure 2. This can be checked via CAYLEY, or in the tables of Neubuser(1967) where BO has the number 48.50.

1.7. Definition: We will denote by NS (for'ninety–six') any group isomorphic to the following soluble group of order 96 :

$$
\begin{aligned}
<a,b,c,d \mid\ & a^2=e , \\
& b^a=b^2 \quad , b^3=1 \quad , \\
& c^a=de^2 \quad , c^b=d \quad , c^2=e^2 \quad , \\
& d^a=ce^2 \quad , d^b=cd \quad , d^c=de^2 \quad , d^2=e^2 , \\
& e^a=e \quad , e^b=e \quad , e^c=e \quad , e^d=e \quad , e^4=1>.
\end{aligned}
$$

The Burnside inclusion diagram of NS is pictured in Figure 3.

2. Subgroups of M_4 .

In this section we discuse some properties of subgroups M_3 and M_4 .

2.1. Group M_4 :

Recall that M_4 is only defined when $p^k \equiv 1 (\mathrm{mod}\ 4)$. We construct a generating set for M_4 in this section by use of preceding methods. Let z be a generator for the scalar group, and define u and v by

$$u:=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \text{ and } v:=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

then {u,v, z} generates the Fitting subgroup of 4 M .To extend this set to a generating set for M_4 .We first require a generating set for $Sp(2,2)$ .We choose this set to consist of the two elements

$$a\rho:=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \text{ and } b\rho:=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

these matrices satisfy the relations

$$(a\rho)^2=I_2,(b\rho)^3=I_2,(b\rho)^{a\rho}=(b\rho)^2.$$

By use from chapter 2 there exist matrices a and b of $GL(2,F)$ satisfying the conditions

$$
\begin{aligned}
u^a &= \lambda_1 v \ , \\
v^a &= \mu_1 u \ , \\
u^b &= \lambda_2 v \ , \\
v^b &= \mu_2 uv.
\end{aligned}
$$

For some scalars λ_1, λ_2, μ_1 and μ_2.

Let w be a primitve **4**-th-root of unity in F, and let δ be an element of F such that

$$\delta^2:=\begin{cases}-2 \text{ if } p^k\equiv 1(\mathrm{mod}\ 8),\\ 2 \text{ if } p^k\equiv 5(\mathrm{mod}\ 8).\end{cases}$$

Setting $\lambda_1=\mu_1=-1$, we find that one solution for a is

$$a:=\delta^{-1}\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}.$$

If $p^k \equiv 1(\mathrm{mod}8)$, then a has determinat 1 and its square is - I_2.

If $p^k \equiv 5(\mathrm{mod}8)$, then a has determinat -ω and its square is ωI_2.

Setting $\lambda_2 = 1$ and $\mu_2 = \omega,$

We find that on solution for b is $b:=2^{-1}\begin{bmatrix}\omega-1 & \omega+1\\ \omega-1 & -\omega-1\end{bmatrix}$

Then b has determinat 1 and order 3, and $b^a = b^2$.

We then the following presentation for M_4 :

$$
\begin{aligned}
\{a,b,u,v,z \mid\ & a^2=\sigma I_2, \\
& b^a=b^2, b^3=I_2, \\
& u^a=-v, u^b=v, u^2=I_2, \\
& v=-u, v^b=\omega uv, v^u=-v, v^2=I_2, \\
& z^a=z, z^b=z, z^u=z, z^v=z, z^{p^k-1}=I_2\},
\end{aligned}
$$

where σ is –1 or w, according as p^k is congruent to 1 or 5 modulo 8 , respectively. Since b dosenot normalise < u,v > , it is more convernient to work < ωu, ωv > Setting $x := \omega\mu$ $y := \omega v$ then we have

$$
\begin{aligned}
\{a,b,x,y,z \mid\ & a^2=\sigma I_2, \\
& b^a=b^2, b^3=I_2, \\
& x^a=-y, x^b=y, x^2=-I_2, \\
& y^a=-x, y^b=xy, y^x=-y, y^2=-I_2, \\
& z^a=z, z^b=z, z^x=z, z^y=z, z^{p^k-1}=I_2\},
\end{aligned}
$$

which is clearly a polycyclic presentation for M_4 (after replacing - I_2 by

$\frac{(p^k-1)}{2}$ and , if necessary, ωI_2 by $z^{\frac{(p^k-1)}{4}}$).

From this repersentation we see that

$$
\begin{aligned}
M_4 &= <z> \mathrm{Y} <a,b,x,y> \\
&= \begin{cases} <z> \mathrm{Y} <a,b,x,y>, & \text{if } p^k\equiv 1(\mathrm{mod}\ 8),\\ <z^4> \times <a,b,x,y>, & \text{if } p^k\equiv 5(\mathrm{mod}\ 8).\end{cases} \\
&\cong \begin{cases} C_{p^k-1}\ \mathrm{Y}\ BO & \text{if } p^k\equiv 1(\mathrm{mod}\ 8),\\ C_{\frac{(p^k-1)}{4}}\times NS & \text{if } p^k\equiv 5(\mathrm{mod}\ 8).\end{cases}
\end{aligned}
$$

It is easy to show that $< a,b, x, y >$ is a minimal supplement to the scalar group. If p^k 5(mod8), k p then it is the unique such supplement.

If $p^k \equiv 1$(mod8), k p then there is just one other minimal supplement to the scalar group; it is$<\omega a,b, x, y >$, which is isomorphic to $GL(2,3)$.

Note that wilson (1972 , theorem 3.2 , p.36) also observed that M_4 Splits over its Fitting subgroup when F has a square root of – 2 (that is, when $p^k \equiv 1$ (mod8)), Also, we have that

$$M_4 \cap SL(2,F) = \begin{cases} < a,b,x,y > & \text{if } p^k \equiv 1(\text{mod } 8) \quad , \\ < b,x,y > & \text{if } \quad p^k \equiv 5(\text{mod } 8). \end{cases}$$

$$\cong \begin{cases} BO & \text{if } p^k \equiv 1(\text{mod } 8) \quad , \\ SL(2,3) & \text{if } p^k \equiv 5(\text{mod } 8) \quad . \end{cases}$$

Finally , we investigate the action of field automorphism on M_4

3.1. Theorem: Every automorphism of F , acting entry-wise on the elements of M_4, normalises M_4.

Proof: Let θ be an automorphism of F. By looking at the entries of the matrices in our generating set for M_4, it is clear that the effect of θ on M_4 is determined by $\omega\theta$ and $\delta\theta$. It is easly to check that , in any case, $x\theta$ is $\pm x$, $y\theta$ is $\pm y$, $a\theta$ is $\pm \omega\theta$ and $b\theta$ is b or by .(and ofcourse $z\theta$ is some power of itself). Therefore θ normalises M_4 .

References

1. Beverley Bolt, TG Room and GE Wall (1961-62) "on the clifford collineation, transform and similarity groups. I and II". j.Aust.Mast.soc 2: 6096.
2. W Burnside (1897) Theory of Groups of Finite Order, 1stedn. Combridge university Press https://www.gutenberg.org/files/40395/40395-pdf.pdf.
3. W Burnside (1911) Theory of Groups of Finite Order, 2nd edn, Cambridge university press. Reprinted by Dover, New York, 1955. https://people.math.rochester.edu/faculty/doug/otherpapers/burnside1911.pdf.
4. Gregory Buttler and John Mckay (1983) "The transitive groups of degree up to eleven". comm Algebra 11: 863-911.
5. John J canon (1984) "An introduction to the group theory language Cayley", in computaional Group Theory, ed.Michael D.Atkinson,Academic press, London 145-183.
6. Jhon Canon (1987) "The subgroup lattice module" in the CAYLEY Bulletin, no.3, ed. John canon, department of pure Mathematics, University of Sydney :42-69.
7. AL Cauchy (1845) CR Acad sci 21: 1363-1369.
8. A Cayley (1891) "On the substitution groups for two, three, four, five, six, seven and eight letters" Quart j pure Appl Math. 25: 137-155.
9. FN Cole (1893b) "The transitive substitution-groups of nine letters" Bull.New York Math soc 2: 250-258.
10. S B Conlon (1977) "Nonabelian subgroups of prime-power order of classical groups of the same prime degree" In group theory, eds R A Bryce, J coosey and M F Newman lecture Notes in Mathematics 573, springer-verlag, Berlin, Heidelberg https://link.springer.com/chapter/10.1007/BFb0087809
11. JH Conway, RT Curtis, SP Norton, RA Parker and R A Wilson (1985) Atlas of Finite Groups, clarendon press, oxford https://global.oup.com/academic/product/atlas-of-finite-groups-9780198531999?cc=us&lang=en&
12. Hyo-Seob Sim (1993) Degree of Irreducible Representations of Metacyclic Groups, J Comm Algebra 21:3773-3777.
13. Charles C Simss (1970) "Computational methods in the study of permutation groups", in computational problems in Abstract Algebra, ed, John Leech, Pergamon press, Oxford 169-183.
14. M Slattery (1986) P i -blocks of P i -separable groups I, j Algebra 102: 60-77.
15. M Slattery (1989) P i -blocks of P i -separable groups II, j Algebra 124: 236-269.
16. M W Short (1992) "The Primitive Soluble Permutation Groups of Degree Less Than 256", Lecture Notes in Mathemaics,1519,springer-verlag Berlin Heidelberg New York https://link.springer.com/book/10.1007/BFb0090195.
17. D A Supruneko (1963) Soluble and Nilpotent Linear Groups. Translation of Mathematical Monographs, vol.9, American Mathematical society, providence, Rhode Island https://www.amazon.com/Soluble-Nilpotent-Translations-Mathematical-Monographs/dp/0821838954
18. DA Suprunenko (1976) Matrix Group, Translation of Mathematical Monographs, VOL.45, American Mathematical society, Provideence, Rhode Island https://www.ams.org/books/mmono/045/
19. B Razzaghmaneshi (2002) Determination of the JS-maximals of general linear group of less than 25, PhD Theses of AUH university.