ISSN: 3033-3652

Journal of Physical Mathematics & its Applications

SCIENTIFIC Research and Community

Review Article

Open Access

Stable Proximal Dynamical System for Mixed Variational Inequalities in Hilbert Spaces

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ABSTRACT

In this paper we focus on solving mixed variational inequalities by proximal fixed-time dynamical systems, in which the solution of the proximal dynamical system is uniquely described and converges to the solution of the associated mixed variational inequality under the assumptions of strong monotonicity and Lipschitz continuity on the operator in the variational inequality, where the time of convergence is finite and is uniformly bounded for all initial conditions. The proposed technique can be reduced to the fixed-time stable projected dynamical system and results can still be applied even with relaxing the assumption of strong monotonicity. The modified continuous-time proximal dynamical system is designed to solve convex minimizing problems on a (possibly infinite-dimensional) Hilbert spaces. Finally, some qualitative properties with fixed-time stability of equilibrium points to the proposed scheme for solving continuous-time nonsmooth convex optimization problems or, in a general setting mixed variational inequalities are presented.

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Received: May 03, 2024; **Accepted:** May 08, 2024; **Published:** August 30, 2024

Keywords: Proximal Dynamical System, Mixed Variational Inequality, Convex Optimization, Fixed-Time Stability, Lyapunov Stability Theory

The Mathematics Subject Classification (MSC): 34H15, 90C25, 93D40.

Introduction

Dynamical systems approaches have been used ubiquitously and become a very fruitful topic of research and powerful alternative tool in developing and advancing techniques for solving mixed variational inequalities (MVIs). MVIs was originally considered by Lescarret and Browder for its applications in mathematical physics. Afterwards, it has been remarkably noted that MVIs have a tremendous variety of real-life applications in several disciplines such as economics and operations research, convex optimization, signal processing, game theory and control theory [1-7]. The dynamical system approach has been used to study the qualitative properties such as, the existence, uniqueness, convergence and stability of the solution of variational inequalities. It was shown, in 1990's, that the variational inequalities can be reformulated in terms of dynamical systems, and their solutions are equivalent to the equilibrium points of the corresponding dynamical systems [6,8]. Among other classes of dynamical systems, we focus our attention in this paper to one essential class of discontinuous dynamical systems, that is, projected dynamical systems whose trajectories remain in a feasible domain by projecting outward portions of a vector field at the boundary of the domain. Projected dynamical systems gain its significance due to its role in the study of variational inequalities and differential inclusions, as well as nonlinear optimization in a more general setting. Hence, this paper is in principle induced by the emerging interest in dynamical systems that solve optimization problems and variational inequalities [6,8-11]. Variational inequality (for short VI), introduced and studied in the early sixties by Stampacchia, is a powerful mathematical model in which it has played a fundamental role in the study of important concepts in equilibrium problems in finance, machine learning and optimization, for instance, the process of minimizing a cost (usually smooth and convex) function can be modeled by a variational inequality, see, for example, Brezis; Cavazzuti; Kinderlehrer and Stampacchia; Noor, et al. Stampacchia, and the references therein [1,5,6,10,12-14]. It is worth mentioning that the variational inequalities are natural generalizations of the variational principles, the origin of which can be traced back to Fermat, Newton, Leibniz, Euler and Largrange. Qualitative properties of VI strongly depend on some kind of monotonicity. In particular, the existence and uniqueness of the solution to the VI can be established under strong monotonicity (strong convexity for real-valued functions), for more details about the methods used in this regard with their variants [15,16]. MVIs simply can be seen as variational inequalities plus a nonlinear term and if the nonlinear term is a proper, convex and lower-semicontinuous function, then one can show that the mixed variational inequalities are equivalent to the fixed point and the resolvent equations [6,8].

Numerous methods for solving MVIs have been considered and developed in the literature which they fall into two main aspects; namely, discrete-time gradient-based algorithms, and continuous-time gradient flows. Several gradient-based descent iterative methods have been analyzed with convergence analysis entails that the operator is strongly monotone and Lipschitz continuous [1,7,15]. Gradient-based algorithms are designed to treat a class of optimization algorithms which are characterized by robustness, Nesterov [17]. The most popular gradient-based is, by Goldstein, the steepest descent approach in which it employs the gradient as search direction along with past gradient information [18]. For accelerating the convergence of these gradient methods, positive definite Hessian matrices are employed by Newton approaches [19]. When the objective function in the optimization problem has sharp edges or discontinuous points (i.e., nonsmooth), then proximal gradient approaches can be employed to access and compute the subgradients [3,4,15]. The most commonly used method for MVI is the proximal point algorithm, and since proximal operators are generalizations of projection operators, it follows that the most commonly used method for variational inequality problems, as a particular case of MVI, is the projection algorithm.

In this paper, we are interested in designing a continuous-time dynamical system such that its solution converges to the solution of the corresponding MVI in a fixed time starting from any given initial condition as it is known that dynamical systems may exhibit dynamics that are highly sensitive to initial conditions. This work is a generalization of Garg et al. from having constrained defined in finite dimensions to the context of infinite dimensional vector spaces [20,21]. The stability analysis of the equilibrium points of the dynamical systems utilizes tools from classical Lyapunov theory. While it is known that the convergence time for a broad class of convex optimization algorithms depends upon the initial conditions and can grow unbounded with the distance of the initial condition from an equilibrium point, the idea of fixed-time stability was introduced in Kinderlehrer D, and used in Nesterov Y, to provide a finite upper bound for all initial conditions [20,21]. To the best of our knowledge, this is the first time a paper proposes fixed-time stable proximal dynamical systems for MVIs, or equivalently, non-smooth convex optimization problems in real Hilbert spaces. In Garg K, modified gradient flow schemes are introduced for unconstrained and constrained convex optimization problems, as well as for minimax saddle problems such that linear equality constraints and continuously differentiable objective function are assumed [22]. In Romero O, the authors proposed dynamical systems as differential inclusions, with solutions converge to strictly local minimizers [23].

The contribution of this paper summarized by two main extensions:
i. Proposing a generalized projection continuous-time dynamical scheme for solving MVIs and can be applied to non-smooth convex optimization problems along with discussing the arguments of existence, uniqueness, convergence and stability;
ii. In the spirit of the ideas introduced by Bello and Hazaimah we extend the work of Garg, et al. to more general variational problems appear in any infinite-dimensional Hilbert space [15,21]. The paper is organized as follows. In section 2 we list some notational backgrounds and essential definitions. In 3, we present a connection between optimization and proximal dynamical systems, and define known systems used in the literature. In section 4, we modify a continuous-time proximal dynamical system based on section 3. Finally, in section 5, we conclude the work and give some future extensions.

Preliminaries

In this section, some optimization-related basics and foundations are presented from monotone operators theory, dynamical systems theory, convex analysis. Let H be a real Hilbert space equipped with inner product $\langle \bullet,\bullet \rangle$ and induced norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. Let $T : H \rightrightarrows H$ be a set-valued (multifunction) operator and its domain be denoted by $\mathbf{dom}(T) := \{x \in H; T(x) < \infty\}$. Let $\mathrm{Fix}(T) := \{x \in H : x = T(x)\}$ be defined as the set of all fixed points of a function T. Define the graph of T by

$$\forall (x,u),(y,v) \in \mathbf{Gph}\,(T), \qquad \langle x-y, u-v\rangle \geq 0.$$

Furthermore, T is maximally monotone if there exists no monotone operator T' such that $\mathbf{Gph}(T')$ properly contains $\mathbf{Gph}(T)$. Recalling that for any maximal operator T (by Brezis,) the *resolvent* operator associated with T is the full domain single-valued operator in H given by $J_T := (I + T)^{-1}$ where $I: H \to H$ denotes the identity operator [3]. The inverse of T is the set-valued operator defined by $T^{-1} : u \mapsto \{x \in \mathcal{H} \mid u \in T(x)\}$. Moreover, $J_{\alpha T} := (I + \alpha T)^{-1}$: $H \to \mathrm{dom}\,(T)$ if $\alpha > 0$. Let C be a nonempty, convex and closed subset of H. The set C is said to be convex set, if $(1-\lambda)x+\lambda y \in C$, $\forall x,y \in C$, $\lambda \in [0,1]$. Assume the function $F : C \to H$, we say F is a convex function, if

$$F((1-\lambda)x + \lambda y) \leq (1-\lambda)F(x) + \lambda F(y),\ \forall x,y \in C,\ \lambda \in [0,1].$$

If the function F is smooth, then the following well known result holds:

Theorem 1: Let C be a nonempty, convex and closed subset of H. Let F be a smooth convex function. Then $x \in C$ is the minimum of the smooth convex $F(x)$ if and only if, $x \in C$ satisfies
$\langle F'(x), y - x\rangle \geq 0, \forall y \in C$
where F' is the Frechet derivative of F at $x \in C$.

This theorem shows that the variational inequalities are natural links and analogous to the minimization of the convex functional subject to certain constraint which has led to study a more general variational inequality. Furthermore, one can define the normal cone operator with respect to $C \subseteq H$ as

$$\mathcal{N}_C(x) := \begin{cases} \varnothing, & \text{if } x \notin C \\ \{y \in \mathcal{H} \mid \langle y, z - x\rangle \leq 0, \quad \forall z \in C\}, & \text{if } x \in C. \end{cases}$$

Hence, the orthogonal projection of x onto C, $\Pi_C(x)$, is given by $\Pi_C(x) = J_{NC}(x) = (I + N_C)(x)$. Now, we state a very well-known fact on orthogonal projections, followed by some useful definitions of monotonocity [24].

Proposition 2 Let C be nonempty closed convex subset of H, and Π_C be the orthogonal projection onto C [25]. For all $x,y \in H$ and all $z \in C$ the following hold:

(i) $\|\Pi_C(x) - \Pi_C(y)\|^2 \leq \|x-y\|^2 - \|(x - \Pi_C(x)) - (y - \Pi_C(y))\|^2$;
(ii) $\langle x - \Pi_C(x), z - \Pi_C(x)\rangle \leq 0$.

Definition 1: Let $C \subset H$ be arbitrary. The operator $T : C \to H$, is called:
(i) monotone, if for all $x,y \in C$,
$$\langle T(x) - T(y), x - y\rangle \geq 0.$$
(ii) strongly monotone if there exists a modulus $\lambda > 0$ such that for all $x,y \in C$,
$$\langle T(x) - T(y), x - y\rangle \geq \lambda\|x - y\|^2.$$
(iii) pseudomonotone, if for all $x,y \in C$,
$$\langle T(y), x - y\rangle \geq 0 \Longrightarrow \langle T(x), x - y\rangle \geq 0.$$
(iv) strongly pseudomonotone if there exists $\lambda > 0$ such that for all $x,y \in C$,
$$\langle T(y), x - y\rangle \geq 0 \Longrightarrow \langle T(x), x - y\rangle \geq \lambda\|x - y\|^2.$$

Notice that the following implications hold, (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii), whereas the converse need not be true generally.

In what follows, we give some notational foundations and significant definitions related to the main model used in this paper, MVIs, which can be formulated by

Find $x^* \in H$ such that $\langle T(x^*), x - x^* \rangle + g(x) - g(x^*) \geq 0$, for all $x \in H$ (1)

where T : **dom** g → H is an operator and $g : H \to \mathbb{R} \cup \{\infty\}$ is a proper (dom g $\neq \emptyset$), lower semi-continuous convex function. If C is a closed and convex set in H and

$$I_C(x) := \begin{cases} 0, & \text{if } x \notin C \\ +\infty, & \text{if } x \in C. \end{cases}$$

is the indicator function of C then problem (1) is reduced to the equivalent and particular case of MVIs, namely, the classical variational inequality (VI), which is equivalent to the generalized equation (a.k.a monotone inclusion)

$$\text{Find } x^* \in \mathcal{H} \quad \text{such that} \quad 0 \in T(x^*) + \partial g(x^*), \qquad (2)$$

where the subdifferential mapping ∂g : H ⇒ H, defined as ∂g(x) := {u ∈ H ; g(y) ≥ g(x)+⟨u,y−x⟩, ∀y ∈ H} is a maximal monotone operator. To extend the inclusion (2), a variety of numerical and analytical methods have been developed for finding a zero of the sum of two operators; one is smooth and another is not (that is, find a point x∗ ∈ H such that 0 ∈ A(x∗) + B(x∗)). One of the most important classical splitting methods to find a zero of the sum A+B is the forward-backward (FB) splitting method introduced in Passty GB which is given as [26].

$$x^{k+1} := J_{\alpha_k B}(x^k - \alpha_k A x^k), \qquad (3)$$

where $\alpha_k > 0$ for all $k \in \mathbb{N}$ with A being smooth and B is nonsmooth. This iteration (3) converges weakly to a point in the solution set of the inclusion $0 \in Ax + Bx$, when the inverse of A is β-strongly monotone (or equivalently A being β-cocoercive), i.e.,

$$\forall x, y \in \mathcal{H}, \qquad \langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2,$$

where $\alpha_k \leq \beta$ for all $k \in \mathbb{N}$ and $\liminf_{k\to\infty} \alpha_k > 0$; see, for instance, [15,27]. It is worth emphasizing that the cocoercivity assumption of an operator is a strictly stronger property than Lipschitz continuity. Recalling that, for some $L \geq 0$, A is L-Lipschitz if

$$\|Ax - Ay\| \leq L\|x - y\| \ , \qquad \forall x, y \in \mathcal{H}.$$

Note that β-cocoercive operators are monotone and $1/\beta$-Lipschitz continuous, but the converse does not hold in general, i.e., there exist monotone and Lipschitz continuous operators that are not cocoercive. If we restrict our choice and take $B = N_C$ the normal cone of C, i.e., find $x \in C$ such that $0 \in A(x) + N_C(x)$, then the MVI can be written as

Find $x \in C$, such that $\langle Ax, y - x \rangle \geq 0$ for all $y \in C$.   (4)

A popular strategy to solve problem (4) relies basically on the cutting plane (a.k.a. localization) idea which is based on finding a suitable hyperplane that separates the solution of the problem from the current iterate and then performs a metric projection step. This kind of idea is used by the famous Korpelevich's *Extragradient method* and its variants for solving problem (4), we refer the reader for more details about screening several methods and survey some recent developments derived from this extragradient one to the work of Bello and Hazaimah and the references therein [4,15]. From now on, unless stated otherwise, we will assume that the operator T is:

(i) Strongly monotone with modulus λ.
(ii) Lipschitz continuous with Lipschitz constant L.
(iii) If T is strongly monotone or strongly pseudomononotone on C ⊆ H, then the variational inequality has at most one solution.

Proximal Dynamical Systems and Optimization

Bridging dynamical systems with optimization has long been studied and developed for the sake of equivalence of the corresponding solutions of optimization problems and equilibrium solutions of dynamical systems [6,8,11,28].

Definition 2: A global continuous time dynamical system is a pair (X,T), where X is a topological space and T : R × X → X is a continuous map so that T(0,x) = x , and T(s,T(t,x)) = T(s + t,x) for all x ∈ X and all t,s ∈ R.
We therefore formulate the main goal of this work as:

Problem 1: Design a continuous-time proximal dynamical system, such that its solution converges to the solution of the MVI (1) within a fixed time, dependent of the initial conditions. Note that the following nonsmooth optimization problem:

$$\min_{x \in \mathcal{H}} f(x) + g(x),$$

with f : **dom**(g) → H being a smooth convex function and g : H → R∪{∞} being a proper, lower semicontinuous convex (not necessarily smooth) function, is equivalent to the MVI if whenever the operator T coincides with the gradient of the smooth function f, namely $T = \nabla f$ in (1). Consider the autonomous (time-invariant) dynamical system:

$$\dot{x} = T(x), \qquad (5)$$

where the vector field $T : H \to H$ is continuous and $T(x^*) = 0$ for some $x^* \in H$.

Definition 3: The equilibrium point x^* of (5) is said to be fixed-time stable if it is Lyapunov stable and

$$\lim_{t \to T(x(0))} x(t) = x^*,$$

where $\sup_{x_0 \in H} T(x_0) < \infty$ and $T : H \to [0,\infty)$.

The following two auxiliary lemmas are significant in proving the main theoretical result of this paper such that they provide properties of Lyapunov function V for fixed-time stability. The existence of such a Lyapunov function for a modified proximal dynamical system lays down the foundation and the analyses for the stability, in which Lemma (3) is used with $\gamma_3 = 1$. However, the proofs are omitted due their existence [20].

Lemma 3: (Lyapunov condition for fixed-time stability). Let $V : H \to [0,\infty)$ be a continuously differentiable unbounded function such that $V(x^*) = 0$, $V(x) > 0$ for all $x \in H \setminus \{x^*\}$ and $V(x) \leq -\left(a_1 V(x)^{\gamma_1} + a_2 V(x)^{\gamma_2}\right)^{\gamma_3}$ with $a_1, a_2, \gamma_1, \gamma_2, \gamma_3 > 0$ such that $\gamma_1\gamma_3 < 1$ and $\gamma_2\gamma_3 > 1$. Then, the equilibrium point x^* of (5) is fixed-time stable with $T(x(0)) \leq \frac{1}{a_1^{\gamma_3}(1-\gamma_1\gamma_3)} + \frac{1}{a_2^{\gamma_3}(\gamma_2\gamma_3 - 1)}$, for any initial condition $x_0 \in H$.

Lemma 4: For any given $\alpha \in (0,1)$, suppose that there is a positive number

$$\epsilon(\alpha) = \frac{\log(\alpha)}{\log(\frac{1-\alpha}{1+\alpha})} > 0.$$

Then,

$$\left(\frac{1-\alpha}{1+\alpha}\right)^{1-\alpha_1} > \alpha, \quad \text{for each} \quad \alpha_1 \in (1-\epsilon(\alpha), 1].$$

Furthermore,

$$\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha_2-1} > \alpha, \quad \text{for each} \quad \alpha_1 \in (1-\epsilon(\alpha), 1].$$

Now, we formally define the most significant mapping in this study, that is the proximal operator. Given the proximal operator associated with a proper lower semi-continuous convex function $f : H \to \mathbb{R} \cup \{\infty\}$ and it is defined as follows:

$$\text{prox}_f(x) := \underset{y\in\mathcal{H}}{\text{argmin}}\, f(y) + \frac{1}{2}\|x-y\|^2.$$

For the purpose of solving the MVI (1), we need the following equivalent result between variational inequalities and the fixed-point problem.

Lemma 5: $C \subseteq H$, and assume the function $x \in C$ is a solution of the variational inequality (4) if and only if $x \in C$ satisfies the relation [14]

$$x = \Pi_C(x - \lambda Tx),$$

where Π_C is the projection operator and $\lambda > 0$ is a constant.

Lemma (5) implies that the variational inequalities (4) is equivalent to the fixed point problem (6). One can define the residue vector $R(x)$ by the relation $R(x) = x - \Pi_C(x - \lambda Tx)$. By Lemma (5), we notice that $x \in C$ is a solution of (4) if and only if $x \in C$ is a zero of the equation $R(x) = 0$. We now consider a projected dynamical system associated with the variational inequalities (4) using the equivalent formulation (6), this class of projected dynamical system was suggested in [6] as

$$\begin{aligned} \dot{x} &= -k\ R(x) \\ &= -k\ \big(x - \Pi_C(x-\lambda Tx)\big), \quad x(t_0) = x_0 \in C, \end{aligned} \tag{7}$$

where k is a parameter. The right hand of (7) is related to the resolvent operator and is discontinuous on the boundary of the set C. In the light of the preceding, and invoking the generalized setting of projected dynamical systems, we now need to consider the following nominal proximal dynamical system:

$$\dot{x} = -k(x - \text{prox}(x - \lambda T(x))), \tag{8}$$

where $k,\lambda > 0$ are some constants. Next lemma connects and establishes the relationship between an equilibrium point of the nominal proximal dynamical system and a solution of the associated MVI.

Lemma 6: A point $x^* \in H$ is an equilibrium point of (8) if and only if it is a solution to the MVI (1).

Proof. From, proposition 12.26), for all $z \in H$ it follows that

$$\begin{aligned} x^* = \text{prox}(x^* - \lambda T(x^*))) &\iff \langle (x^* - \lambda T(x^*)) - x^*, z - x^*\rangle + \lambda g(x^*) \le \lambda g(z) \\ &\iff \lambda\langle T(x^*), z - x^*\rangle + \lambda g(z) - \lambda g(x^*) \ge 0 \\ &\iff \langle T(x^*), z - x^*\rangle + g(z) - g(x^*) \ge 0. \end{aligned}$$

Hence, the conclusion applies [2].

Next lemma is very important and auxiliary for showing the contribution and efficiency of the main result of this paper.

Lemma 7: Given $\lambda > 0$ and its upper bound is $\frac{2\mu}{L^2}$, *let* $\alpha = \sqrt{1 - 2\lambda\mu + \lambda^2 L^2} \in (0,1)$. Then for all $x \in H$, we have

$$\|\ prox_{\lambda g}(x - \lambda T(x)) - x^*\| \le \alpha\|x - x^*\|,$$

provided that x^* is an equilibrium point of (8).

Proof: Based on Proposition 26.16 (ii)), it is known that for a proper, lower semi-continuous convex function g, the proximal operator prox is ***firmly nonexpansive*** [25]. Hence, the following inequality:

$$\begin{aligned} \|\text{prox}(x - \lambda T(x)) - \text{prox}(x^* - \lambda T(x^*))\|^2 &\le \|(x - \lambda T(x)) - (x^* - \lambda T(x^*))\|^2 \\ &= \|x - x^*\|^2 - 2\lambda\langle T(x) - T(x^*), x - x^*\rangle \\ &\quad + \lambda^2\|T(x) - T(x^*)\|^2 \end{aligned} \tag{9}$$

holds for all $x \in H$. Using Lemma 6, it follows that $x \in H$ is also an equilibrium point of (8) and therefore, $x^* = \text{prox}(x^* - \lambda T(x^*)))$. This result along with the assumption that the operator F is strongly monotone and Lipschitz continuous, further implies that the right hand side of (9) can have an upper bound such that:

$$\begin{aligned} \|\text{prox}(x - \lambda T(x)) - \text{prox}(x^* - \lambda T(x^*))\|^2 &\le \|x - x^*\|^2 - 2\lambda\mu\|x - x^*\|^2 \\ &\quad + \lambda^2 L^2\|x - x^*\|^2, \\ &= (1 - 2\lambda\mu + \lambda^2 L^2)\|x - x^*\|^2. \end{aligned} \tag{10}$$

Thus, the result concludes by taking the square root of both sides of the inequality (10), where $\alpha = \sqrt{1 - 2\lambda\mu + \lambda^2 L^2}$ and $\lambda \in (0, 1)$. Recall that an operator F is non expansive if it is Lipschitz with constant 1. Moreover, F is firmly nonexpansive if

$$\forall x, y \in \mathcal{H}, \quad \|T(x) - T(y)\|^2 \le \|x - y\|^2 - \|(x - T(x)) - (y - T(y))\|^2.$$

Modified Proximal Dynamical System

In the following, we introduce the modified proximal dynamical system, in which its equilibrium point is fixed-time stable and the coefficients are no longer exist (that is, the coefficient in the right-hand side of the differential equation depends on the state x). Consider the modification of (8) given by:

$$\dot{x} = -k(x)(x - \text{prox}(x - \lambda T(x))), \tag{11}$$

where

$$k(x) = \begin{cases} 0 & \text{if } x \in \textbf{Fix}\,(\text{prox}) \\ \dfrac{k_1}{\|x - \text{prox}(x - \lambda Tx)\|^{(1-\alpha_1)}} + \dfrac{k_2}{\|x - \text{prox}(x - \lambda Tx)\|^{(1-\alpha_2)}} & \text{if } x \in \mathcal{H} \setminus \textbf{Fix}\,(\text{prox}) \end{cases}$$

with $k_1, k_2 > 0$, $\alpha_1 \in (0,1)$ and $\alpha_2 > 1$. For the points outside the set of the fixed points of the proximal operator, the first fraction in the piece wise function k above has the finite-time stability of the equilibrium point of (11), whereas the second term converges to the equilibrium point of (11) uniformly for any given condition and constructs an upper bound for the convergence rate [22]. Next lemma institutes and confirms the equivalence between equilibrium points of the modified and nominal proximal dynamical systems.

Lemma 8: A point x∗ ∈ H is an equilibrium point of (11) if and only if it is an equilibrium point of (8).
Proof. It is obvious that if x^* ∈ H is an equilibrium point of (11), i.e., x∗ ∈ **Fix**(prox), and by using the piecewise k(x) then we can see that x∗ is also an equilibrium point of (8). For the other direction, we just need to observe that for any equilibrium point x ∈ **Fix**(prox), we have k(x) = 0.

Remark 1: If the vector field G in (14) is selected as the one in (8), i.e., $G(x) := x−\text{prox}(x−\lambda F(x))$ for any $x \in$ H, then G satisfies the inequality $\langle x−x*,G(x)\rangle > 0$, $\forall x \in H\backslash\{x^*\}$, where x∗ is the solution of the MVI (1), i.e., **Fix**(prox) has only a unique element $x^* = \bar{x}$. Furthermore, and for all $x \in$ H, the following equality holds,

$$\langle x - x^*, x - \text{prox}(x - \lambda F(x))\rangle = \|x - x^*\|^2 + \langle x - x^*, x^* - \text{prox}(x - \lambda F(x))\rangle.$$

Using the Cauchy-Schwarz inequality and lemma (7), then (13) has a lower bound as:

$$\langle x - x^*, x - \text{prox}(x - \lambda F(x))\rangle \geq (1-\alpha)\|x - x^*\|^2,$$

where $\alpha \in (0,1)$, which implies that $\langle x - x^*, G(x)\rangle > 0$ for all $x \in H \setminus \{x^*\}$.

The following proposition assures that the solutions of (11) exist and are uniquely determined for all future iterations.

Proposition 9: Let G : H → H be a locally Lipschitz continuous vector field such that

$$G(\bar{x}) = 0 \quad and \quad \langle x - \bar{x}, G(x)\rangle > 0$$

for all $\bar{x} \in H \setminus x$,where $\bar{x} \in$ H. Consider the following autonomous dynamical system:

$$\dot{x} = -\sigma(x)G(x) \tag{14}$$

where

$$\sigma(x) = \begin{cases} 0, & if\ G(x) = 0; \\ \dfrac{k_1}{\|G(x)\|^{(1-\alpha_1)}} + \dfrac{k_2}{\|G(x)\|^{(1-\alpha_2)}}, & otherwise, \end{cases}$$

with $k_1,k_2 > 0$, $\alpha_1 \in (0,1)$ and $\alpha_2 > 1$. Then, the right hand side of (14) is continuous for all $x \in$ H, and there exist a solution of (14), wich is uniquely determined for all $t \geq 0$ and for any initial condition.

Proof: We first treat the existence claim followed by the uniqueness of a solution. Since the piece wise function σ is continuous on all points belong to the set $H\backslash\{x^*\}$ and the vector field G is locally Lipschitz continuous on H, then G is continuous at x∗ ∈ H. Note that $\lim_{x\to x*} \sigma(x)G(x) = 0$, since $\alpha_1 \in (0,1)$ and $\alpha_2 > 1$. For any given initial point, the equilibrium point $x*$ ∈ H of the vector field G can be shown that it is globally asymptotically stable and hence, it is unique. Using the fact that G is continuous, it follows from Scutari G, that for any given x_0 ∈ H, there exists a solution of (14) on some interval $[0,\tau(x_0)]$, with $\tau(x_0) > 0$. Moreover, by Scutari G, the maximal interval of existence for any such solution of (14) is $[0,\bar{\tau}(x_0))$. Thanks to the stability theory of dynamical systems such that we can assume the function V : H → [0,∞) defined as $V(x) := \frac{1}{2}\|x - x^*\|^2$, to be the Lyapunov function for the dynamical system (14) along with its trajectory (i.e., solution curve), for any x_0 ∈ H, written as [29].

$$\dot{V}(x(t)) = -\langle x(t) - x^*, \sigma(x(t))G(x(t))\rangle,$$

which implies $\dot{V}(x(t)) \leq 0$, since $\sigma(x) \geq 0$, $\forall\ x \in H$ and by using remark (1), for all $t \in [0, \bar{\tau}(x_0))$, consequently, $V(x(t)) \leq V(x_0)$ and any solution of (14) defined on $[0,\bar{\tau}(x_0))$ lies entirely in the set $K_{x0} := \{z \in H ; \|z - x^*\| \leq \|x_0 - x^*\|\}$. It follows that, by Stampacchia G the solution goes to infinity inside the compact set K_{x0} (i.e., $\tau(x_0) = \infty$), thus, it is not everywhere defined and this completes the existence argument [30]. Next we discuss the uniqueness, for any given $x_0 \in H$, let x^s be a solution of (14) with the initial condition $x^{sol}(0) = x_0$. To this end, we need to consider two cases related to the equilibrium point of the dynamical system (14); (i) $x^{sol}(0) = x^*$, and (ii) $x_{sol}(0) \in$ H \ $\{x^*\}$. In the first case, we consider the same Lyapunov function V above and following the same guidelines from the existence argument we would have come to the conclusion that the solution of (14) is non-positive for any initial point. Hence, xsol is uniquely determined. Let T := $\inf\{t \geq 0 : x^{sol}(t) = x^*\}$ which is strictly positive by the continuity of x^{sol}. Next, consider the parameterization Φ: $[0,T) \to [0,\infty)$ defined as follows:

$$\Phi(t) := \int_0^t \sigma(x^{sol}(v))\, dv \tag{15}$$

Since the integrand $\sigma(x^{sol}(v))$ is continuous on H and strictly positive for any $v \in [0,T)$, and since x^{sol} is continuous on $[0,T)$, it follows that the function Φ is a strictly increasing continuous function, for all $t \in (0,T)$. Furthermore, from the inverse function Theorem, it follows that Φ^{-1} exists and strictly increasing continuous. Hence, a solution corresponding to the vector field in (14) is also a solution corresponding to the vector field G, under the parameterization (15). Furthermore, since the vector field G is locally Lipschitz continuous on H, it can be shown that for any given initial condition, there exists a unique solution corresponding to G by following similar steps of the existence argument [21]. Hence, $\bar{x}^{sol}$ is uniquely determined and since the function Φ is injective, with Φ(0) = 0, it follows that x^{sol} is also uniquely determined.

The following theorem establishes the first main result of the paper.

Theorem 10: For any given
$\lambda \in (0, \frac{2\mu}{L^2})$, *let* $\alpha = \sqrt{1 - 2\lambda\mu + \lambda^2 L^2} \in (0,1)$ and

$$\epsilon(\alpha) = \frac{\log(\alpha)}{\log(\frac{1-\alpha}{1+\alpha})} > 0.$$

Then, the solution $\bar{x} \in H$ of (1) is a fixed time stable equilibrium point of (11) for any $\alpha_1 \in (1-\epsilon(\alpha),1) \cap (0,1)$ and $\alpha_2 \in (1,1+\epsilon(\alpha))$.

Proof: By (Proposition 12.28), it follows that the vector field in (8) is Lipschitz continuous on H, with a unique equilibrium point and satisfies Proposition (9) (see Remark (1)) [2]. Hence, from Proposition (9), it follows that starting from any initial condition, a solution of (11) exists and is uniquely determined.

Consider now an unbounded Lyapunov function $V : H \to [0,\infty)$ defined as $V(x) := \frac{1}{2}\{x - \bar{x}\|^2$, where, from Lemma 5, $\bar{x} \in H$ is the unique equilibrium point of the vector field in (11). The time-derivative of V along the solution of (11), starting from any $x_0 \in H \setminus \{\bar{x}\}$, reads:

$$\begin{aligned}\dot{V} &= -\Big\langle x-\bar{x}, k_1\frac{x-\text{prox}(x-\lambda T(x))}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_1)}} + k_2\frac{x-\text{prox}(x-\lambda T(x))}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_2)}}\Big\rangle \\ &= -\Big\langle x-\bar{x}, k_1\frac{x-\bar{x}}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_1)}} + k_2\frac{x-\bar{x}}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_2)}}\Big\rangle \\ &- \Big\langle x-\bar{x}, k_1\frac{\bar{x}-\text{prox}(x-\lambda T(x))}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_1)}} + k_2\frac{\bar{x}-\text{prox}(x-\lambda T(x))}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_2)}}\Big\rangle\end{aligned}$$

By applying the Cauchy–Schwarz inequality on the second term of the right hand side of the above, we have

$$\begin{aligned}\dot{V} \leq &-\left(k_1\frac{\|x-\bar{x}\|^2}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_1)}} + k_2\frac{\|x-\bar{x}\|^2}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_2)}}\right) \\ &+\left(k_1\frac{\|x-\bar{x}\|\|\bar{x}-\text{prox}(x-\lambda T(x))\|}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_1)}} + k_2\frac{\|x-\bar{x}\|\|\bar{x}-\text{prox}(x-\lambda T(x))\|}{\|x-\text{prox}(x-\lambda T(x))\|^{(1-\alpha_2)}}\right)\end{aligned} \tag{16}$$

Now be revoking lemma (7) and by using the triangle inequality, since $\lambda \in (0, \frac{2\mu}{L^2})$, there exists $\alpha \in (0,1)$ such that the following inequality:

$$\|x-\text{prox}(x-\lambda T(x))\| \leq \|x-\bar{x}\| + \|\text{prox}(x-\lambda T(x))-\bar{x}\| \leq (1+\alpha)\|x-\bar{x}\|$$

holds for all $x \in H$. Similarly, by using the reverse triangle inequality, there exists $\alpha \in (0,1)$ such that the following inequality:

$$\|x-\text{prox}(x-\lambda T(x))\| \geq \|x-\bar{x}\| - \|\text{prox}(x-\lambda T(x))-\bar{x}\| \geq (1-\alpha)\|x-\bar{x}\|$$

also holds for all $x \in H$. Using the last two inequalities and Lemma (7), then the inequality (16) will have the following upper bound:

$$\begin{aligned}\dot{V} \leq &-\left(\frac{k_1\ \|x-\bar{x}\|^{1+\alpha_1}}{(1+\alpha)^{1-\alpha_1}} + \frac{k_2\ \|x-\bar{x}\|^{1+\alpha_2}}{(1-\alpha)^{1-\alpha_2}}\right) + \left(\frac{\alpha k_1\ \|x-\bar{x}\|^{1+\alpha_1}}{(1-\alpha)^{1-\alpha_1}} + \frac{\alpha k_2\ \|x-\bar{x}\|^{1+\alpha_2}}{(1+\alpha)^{1-\alpha_2}}\right) \\ =&\left(\frac{\alpha}{(1-\alpha)^{1-\alpha_1}} - \frac{1}{(1+\alpha)^{1-\alpha_1}}\right)k_1\ \|x-\bar{x}\|^{1+\alpha_1} + \left(\frac{\alpha}{(1+\alpha)^{1-\alpha_2}} - \frac{1}{(1-\alpha)^{1-\alpha_2}}\right)k_2\ \|x-\bar{x}\|^{1+\alpha_2} \\ =&-(1-\alpha)^{\alpha_1-1}\left[\left(\frac{1-\alpha}{1+\alpha}\right)^{1-\alpha_1} - \alpha\right]k_1\ \|x-\bar{x}\|^{1+\alpha_1} \\ &-(1+\alpha)^{\alpha_2-1}\left[\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha_2-1} - \alpha\right]k_2\ \|x-\bar{x}\|^{1+\alpha_2}.\end{aligned} \tag{17}$$

From Lemma 2, it follows that there exists $\epsilon(\alpha) = \dfrac{\log(\alpha)}{\log\left(\dfrac{1-\alpha}{1+\alpha}\right)} > 0$ for any $\alpha_1 \in (1-\epsilon(\alpha), 1) \cap (0,1)$

and $\alpha_2 \in (1, 1 + \epsilon(\alpha))$. Hence, the proof can be concluded using Lemma 1.

In the special case, when the function w in (3) is chosen to be the indicator function of a non-empty, closed convex set $C \subseteq H$, the proximal operator reduces to the projection operator, i.e., $\Pi_C = \text{prox}_{IC}$, where the projection operator is defined as

$$\Pi_C = \operatorname*{argmin}_{y\in C}\|x-y\|.$$

Thus, the proximal dynamical system reduces to a symbolic projected dynamical system:

$$\dot{x} = -k(x - \Pi_C(x-\lambda F(x))),$$

with $k,\lambda > 0$, which can be used to solve variational inequalities [1,31]. Furthermore, the modified proximal dynamical system now reduces to a modified projected dynamical system:

$$\dot{x} = -k(x)(x - \Pi_C(x-\lambda F(x))),$$

It is shown in Cavazzuti E, et al. and Xia Y, et al. that the equilibrium point of (19) is globally exponentially stable for a strongly pseudomonotone and Lipschitz continuous operator F [1,31]. To establish the fixed-time stability of the equilibrium point of the modified projected dynamical system, we need to consider the following corollary, which is a subsequent of Theorem 10.

Corollary 11: For any given λ > 0 with its upper bound is $\frac{2\mu}{L^2}$, there exist $\alpha \in (0,1)$ and $\epsilon(\alpha > 0)$ as given in Theorem 10, such that the solution $x \in C$ in (1), with $g = I_C$, where C is a closed convex set, is a fixed-time stable equilibrium point of (4) for any $\alpha_1 \in (1 - \epsilon(\alpha),1) \cap (0,1)$ and $\alpha 2 \in (1,1 + \epsilon(\alpha))$.

Remark 2: In dealing with indicator functions rather than general functions, then we would be restricted to a particular case, the case of projection operators. In such sense, lemma (7), and corollary (11) remain valid even if the assumption of strong monotonocity may be relaxed to that of strong pseudomonotonicity which is a special case of corollary (11).

Conclusion and Extensions

In Hilbert spaces, convexity on functions and global Lipschitz continuity on the gradients are sufficient for providing convergence of the sequence generated. Continuous-time dynamical systems propose dynamic visions into designing consistent schemes for solving unconstrained optimization problems in Hilbert space and their equivalence class MVIs. This paper is an extension of the work of the authors in Garg K into infinitedimensional variational problems, in which the solution (the equilibrium point) of the modified proximal dynamical system converges to the unique solution of the associated MVIP in a fixed time, under the assumptions of strong monotonicity and Lipschitz continuity on the associated operator [21]. Furthermore, the proposed proximal dynamical system reduces to a fixed-time stable projected dynamical system, where the fixed-time stability of the modified projected dynamical system continues to hold, even if the assumption of strong monotonicity is relaxed to that of strong pseudomonotonicity. One suggestion to extend this work is by relaxing Lipschitz continuity. Also, the strong monotonocity assumption can be relaxed to the monotonicity case, like wise, to the more general class of pseudomonotone operators.

Another direction of future research, which has a promising work in the practical scope, is by applying forward-Euler discretization of the modified proximal dynamical system explicit discrete-time approximation scheme. However, a robust discrete-time approximation scheme must be chosen for the generated sequence such that it preserves the convergence behavior of the continuous-time dynamical system because in general the fixed-time convergence cannot be preserved. Finally, this work could be generalized to the mixed equilibrium problems or mixed quasivariational inequalities and we predict the qualitative results to still hold with possibly careful observation and much work especially in the aspect of globally asymptotically or exponentially stable.

Declarations

The author declares that there was no conflict of interest or competing interest.

References

1. Cavazzuti E, Pappalardo M, Passacantando M (2002) Nash equilibria variational inequalities and dynamical systems. Journal of Optimization Theory and Applications 114: 491-506.
2. Facchinei F, Pang JS (2003) Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer https://link.springer.com/book/10.1007/b97543.
3. Giannessi F, Maugeri A, Pardalos P M (2001) Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models. Springer https://catalog.lib.msu.edu/Record/hlm.ebs319543e.
4. Korpelevich GM (1976) The extragradient method for finding saddle points and other problems. Ekonomika i Matematcheskie Metody 12: 747-756.
5. Noor MA (1990) Mixed variational inequalities. Applied Mathematics Letters 3: 73-75.
6. Noor M A, Noor K I, Latif R (2017) Dynamical Systems and Variational Inequalities. Journal of Inequalities and Special Functions 8: 22-29.
7. Scutari G, Palomar D P, Facchinei F, Pang JS (2010) Convex optimization, game theory and variational inequality theory. IEEE Signal Processing Magazine 27: 35-49.
8. Dupuis P, Nagurney A (1993) Dynamical Systems and Variational Inequalities. Annals of Operations Research 44: 9-42.
9. Hauswirth A, Bolognani S, Dorfler F (2021) Projected Dynamical Systems on Irregular, NonEuclidean Domains for Nonlinear Optimization. SIAM Journal on Control and Optimization 59: 635-668.
10. Brezis H (1973) Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Publishing Co., Amsterdam-London, American Elsevier Publishing Co. Inc., New York https://www.scirp.org/reference/referencespapers?referenceid=2877183.
11. Pappalardo M, Passacantando M (2002) Stability for Equilibrium Problems: From Variational Inequalities to Dynamical Systems. Journal of Optimization Theory and Applications 113: 567-582.
12. Stampacchia G (1964) Formes bilineaires coercitives sur les ensembles convexes. Comptes Rendus Acad Sci Paris 258: 4413-4416.
13. Kinderlehrer D, Stampacchia G (2000) An Introduction to Variational Inequalities and Their Applications. SIAM https://www.scirp.org/reference/referencespapers?referenceid=968405.
14. Noor M A (2004) Some developments in general variational inequalities. Appl math Comput 251: 199-277.
15. Bello-Cruz Y, Hazaimah O (2022) On the weak and strong convergence of modified forward-backwardhalf-forward splitting methods. Optimization Letters 17: 1-23.
16. Bello-Cruz Y, Oliveira W (2016) On weak and strong convergence of the projected gradient method for convex optimization real in Hilbert spaces. Numerical Functional Analysis and Optimization 37: 129-144.
17. Nesterov Y (2013) Introductory lectures on convex optimization: A basic course. Springer Science and Business Media https://pages.cs.wisc.edu/~yliang/cs839_spring22/material/Introductory-Lectures-on-Convex-Programming-Yurii-Nesterov-2004.pdf.
18. Goldstein A (1965) On steepest descent. Journal of the Society for Industrial and Applied Mathematics, Series A: Control 3: 147-151.
19. Alvarez F, Bolte J, Olivier B (2004) Hessian Riemannian gradient flows in convex programming. SIAM Journal on Control and Optimization 43: 477-501.
20. Polyakov A (2012) Nonlinear feedback design for fixed-time stabilization of linear control systems. IEEE Transactions on Automatic Control 57: 2106-2110.
21. Garg K, Baranwal M, Gupta R, Benosman M (2022) Fixed-

Time Stable Proximal Dynamical System for Solving MVIPs. IEEE Transactions on Automatic Control 68: 5029-5036.

22. Garg K, Panagou D (2021) Fixed-Time stable gradient flows: Applications to continuous-time optimization. IEEE Transactions on Automatic Control 66: 2002-2015.
23. Romero O, Benosman M (2020) Finite-Time convergence in continuous-time optimization. In Proceedings of the International Conference on Machine Learning 119: 8200-8209.
24. Karamardian S, Schaible S (1990) Seven kinds of monotone maps. Journal of Optimization Theory and Applications 66: 37-46.
25. Bauschke HH, Combettes PL (2017) Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, New York https://pcombet.math.ncsu.edu/livre1.pdf.
26. Douglas J, Rachford H H (1956) On the numerical solution of heat conduction problems in two or three space variables. Transactions of the American Mathematical Society 82: 421-439.
27. Passty GB (1979) Ergodic convergence to a zero of the sums of monotone operators in Hilbert space. Journal of Mathematical Analysis and Applications 72: 383-390.
28. Friesz TL (2010) Dynamic Optimization and Differential Games. International Series in Operations Research & Management Science 135: 138-142.
29. Hale JK (1980) Ordinary Differential Equations. Krieger Publishing Company https://www.scirp.org/reference/referencespapers?referenceid=953267.
30. Bhat SP, Bernstein DS (2000) Finite-Time stability of continuous autonomous systems. SIAM Journal on Control and Optimization 38: 751-766.
31. Xia Y, Wang J (1998) A general methodology for designing globally convergent optimization neural networks. IEEE Transactions on Neural Networks 9: 1331-1343.