ISSN: 2754-4753

Journal of Physics & Optics Sciences

SCIENTIFIC Research and Community

Review Article Open Access

Studying on a Computer the Movement of a Light Beam inside a Mirror Ellipse

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ABSTRACT
An exact formula has been found (on a computer) that predicts the n-th reflection of a light ray in a mirror ellipse. It has a paradoxical character.

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Received: June 01, 2024; **Accepted:** June 24, 2024; **Published:** June 30, 2024

Introduction

Let's take the movement of a ray of light along a mirror circle (a=1, b=1: a and b are semi-axes along the X and Y axes). This is completely trivial and does not cause any interest! The reflection point simply rotates at a certain angle each time.

Now we stretch the circle along the X axis, take a>1, the situation will change! Let's immediately introduce 7 new(?) mathematical facts that we found on the computer. They include, among other things, four numerical invariants associated with the tandem "mirror ellipse + an infinite sequence of reflections of a light ray in it" and two exact formulas. Moreover, the main "**Formula number 2**" will have an amazing character.

What can we tell at first glance by looking at the reflections of light in an ellipse?

Obviously, there are two types of movement of a light ray in an ellipse: those whose rays do not intersect the segment between the foci (all), and those that do (also all). See Figure 1. This statement is probably not difficult to prove.

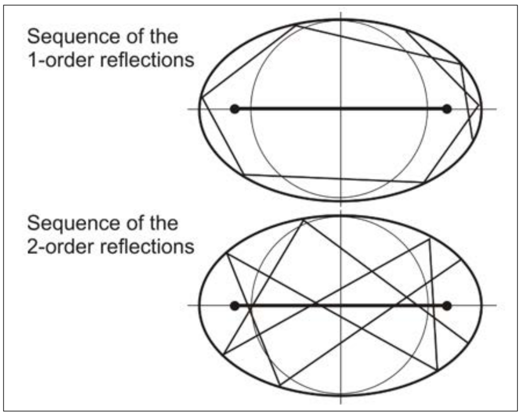

Figure 1: Two possible types of movement of a Light Beam inside an Ellipse

Another statement follows from observations. For any non-closed (infinite) sequence of reflections, there is always a reflection point that is located at an arbitrarily small distance from point C (and, similarly, from point D). See Figure 2.

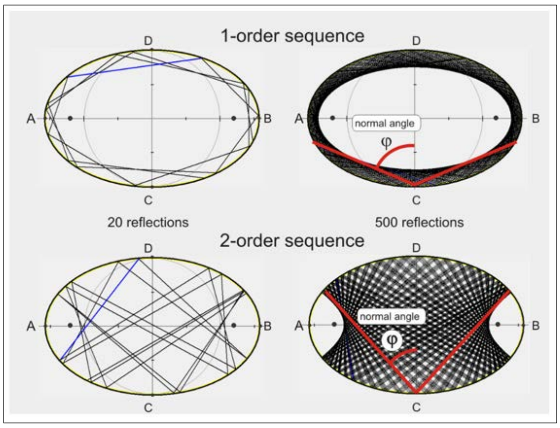

Figure 2: Normal angle (j) for a sequence of Reflections

Let's call half the reflection angle (for the limiting case) at this point – φ, the *normal angle* for this sequence and call this (unproven) statement a *normal hypothesis*.

The normal angle also exists for finite sequences (Figure 3a).

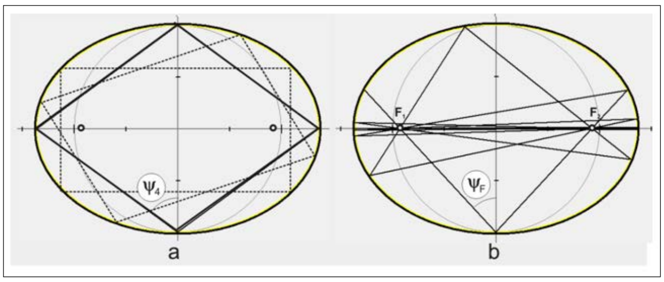

Figure 3a: Let's add a small angle to ψ_4 and the quadrilateral

(the original rhombus) will begin to "travel along the ellipse", changing, passing through the "rectangle phase" along the way. That is, the normal angle of the rectangle exists and is equal to $\psi 4$; 3b: normal angle ψ_F separates sequences of 1^{st} and 2^{nd} order (for $\varphi > \psi 4$ the movement is 1st order, for $\varphi < \psi_F$ – 2nd).

The First Six Undeniable Facts about Reflections

Let us list, separated by commas, SIX new mathematical facts that relate to sequences of reflections in mirror ellipses (or in mirror *ellipsoid of revolution* in R^N). See Figure 4-5. (We will present the seventh fact – "**Formula Number 2**" – in a new chapter).

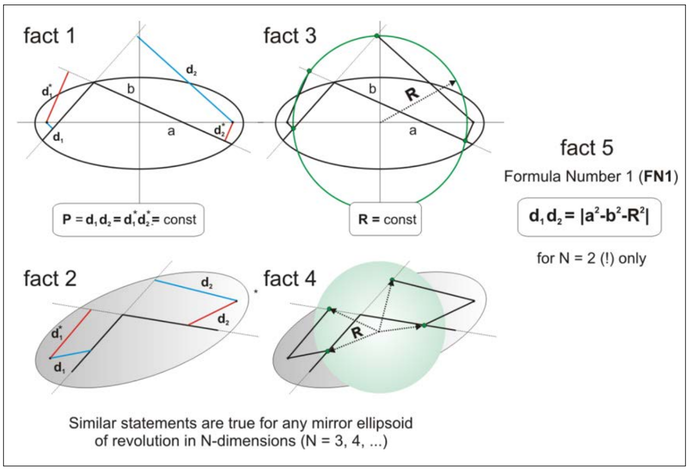

Figure 4: Our first five facts about the Reflection of rays in a Mirror Ellipse (and N-dimensional mirror *ellipsoid of revolution*)

Let's list...

Fact 1: The product of the distances from the foci to the rays of the sequence is a constant for any ray. (Our first invariant is the product $P=d_1d_2$).

Fact 2: The previous statement is also true for any mirror ellipsoid of revolution in RN (tested N= 3, 4...).

Fact 3: The bases of the perpendiculars that are drawn from the foci to the rays lie on the same circle (let's call it the *base* circle) of radius R with the center at the center of the ellipse. R is the radius of the base circle – our second invariant.

Fact 4: The previous statement is also true for any mirror ellipsoid of revolution in R^N. (R is the radius of the N-dimensional ball in RN; tested N = 3, 4...)

Fact 5: how the product P=d1d2 and the base radius R are related in two dimension?

$$d_1 d_2 = \left| a^2 - b^2 - R^2 \right| \qquad (1)$$

See Figure 4, fact 6.

Now in reference books and literature only facts No. 1 and 5 are given, and in the version for tangents [1, 2]. In the reference book they sound like this: "for the flat ellipse, the product $P=d_1d_2$ for all tangents is the same and equal to b^2." Although these statements are a simple transition to the limit of our statements (facts 1 and 5) as the beam length tends to zero.

All invariants – both the product $P=d_1d_2$ and the base radius R – are determined simply from the appearance of the initial ray! The normal angle can also be determined from it. For angle q=p/2-j we have a system of equations $sin^2(q)(x^2-c^2)=P$; $b/x=tg(q)$; $(c^2=a^2-b^2)$ => $x= cos(q)*b/sin(q)$ => $sin^2(q)= b^2/(b^2- c^2+P)$.

Now comes the sixth remarkable fact (see Figure 5).

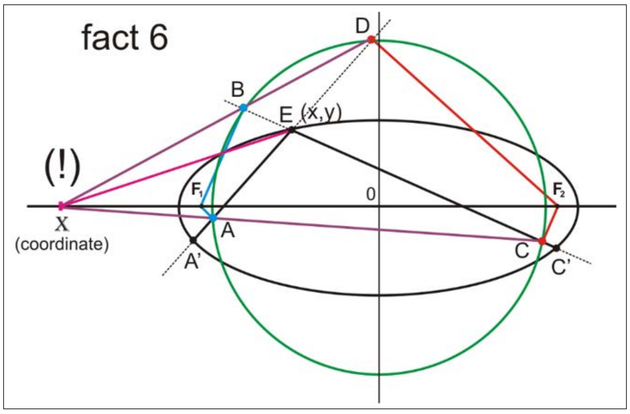

Figure 5: If the sequence of reflection points on the ellipse is equal to A'EC', then the lines AC and BD will intersect at the point with coordinates (X,0). In reverse order - C'EA' - the intersection point will remain the same. (Direct AC and BD will simply switch places).

Thus, any sequence of reflections $\{x_n,y_n\}$ can be translated into a one-dimensional sequence $\{X_n\}$.

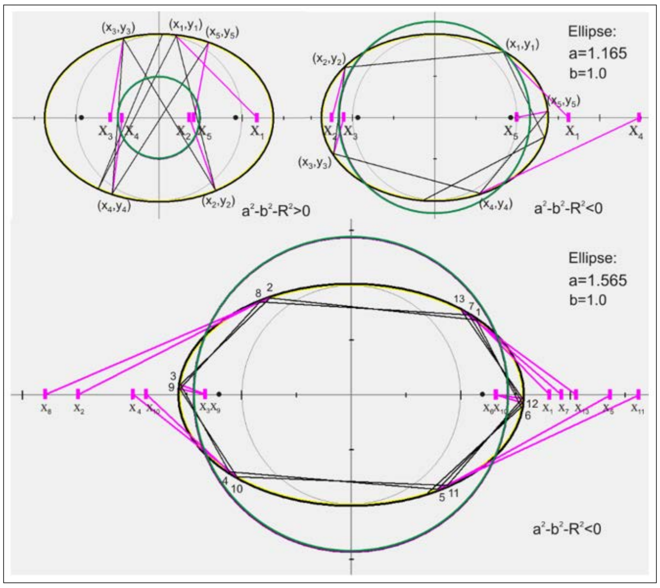

Figure 6: Examples of transforming reflection points on an ellipse (x,y) into a one-dimensional representation {X}

Amazing Formula Number 2 (FN2)

Recall that the reflection operation is reversible, and we can reflect any ray both forward and backward. This circumstance should obviously be extended to the sequence $\{X_n\}$.

The book "Cellular Automaton Machines" by Toffoli T. and Margolus N presents a standard formula for one-dimensional real reversible automata (sequences) [3]:

$$X_{n+1} = F(X_n) - X_{n-1} \qquad (1)$$

where F is an arbitrary function.

Let's assume that our sequences X_n also satisfy formula (1) with some function F, and let's try to determine it.

Let's take a specific sequence of reflections in an arbitrary ellipse and construct a two-dimensional graph *P*: $\{X_n \times X_{n-1}+X_{n+1}\}$. Results in Figure 7.

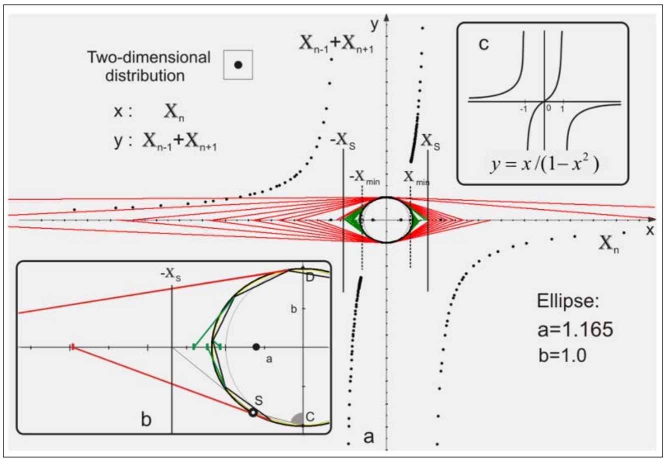

Figure 7a: The figure simultaneously shows both a one-dimensional representation of X_n (for part of the sequence) and a plot {Xn x Xn+1 + Xn-1}. Two singularity points (indicated by vertical solid lines) with coordinates in one-dimensional representation –XS and +XS are clearly visible on the plot. Green straight lines and small rectangles for one-dimensional representation indicate those points for which |Xn|<XS; red – those for which |Xn|>XS. The dotted vertical lines indicate the boundaries of the interval [$-X_{min}$, X_{min}] into which the points of the one-dimensional representation do not fall at all! (A point on an ellipse with coordinates (a,0) goes, in the limit, to the coordinate of a one-dimensional representation X_{min}; similarly: a point (-a,0) goes to $-X_{min}$). Figure 7b is an enlarged view of our ellipse for the first few reflections. Point S is a "singularity point" on the ellipse (one of four; transitions in one-dimensional representation to -XS). The *SCD* angle is the same as the normal angle for this sequence.

It can be seen that the distribution of points on the plot is very similar to the function presented in Figure 7c.

Let's write it like this:

$$X_{n+1} = kX_n / (X_S^2 - X_n^2) - X_{n-1} \qquad (2)$$

Using the first two reflections (four numbers X_1, X_2, X_3, X_4), we write down a system of two unknowns and determine *k* and X_S respectively.
We get

$$X_S^2 = (gX_2^2 - X_3^2)/(g-1) \qquad (3)$$

where

$$g = (X_3X_1 + X_3^2)/(X_2X_4 + X_4^2) \qquad (4)$$

And for the second unknown:

$$k = (X_S^2 - X_2^2)(X_1 + X_3)/X_2 \qquad (5)$$

This set of formulas (2-5) has our common name: Formula Number 2 (**FN2**) where the numbers *k* and X_S – this is our third and fourth Invariant. (There are four invariants in total: *P, R, k* and X_S).

To check *FN2* it is necessary to compare for each number n (Figure 7) two quantities: $X_{n+1}+X_{n-1}$ and its approximation $kX_n/(X_S^2+X_n^2)$. But both of them are EXACTLY designed for us! This means that the error we found is just a computer error! It turns out that the two indicated values for all points of the plot always coincide up to the sixth to eighth digit, and this is for any ellipse and for any initial ray!

Let us give a countable example for our new invariants. They must also be matched: X_S^2 and k built on the numbers X_1, X_2, X_3, X_4; on the numbers X_2, X_3, X_4, X_5; on the numbers X_3, X_4, X_5, X_6 etc.

Let us present the first five XS2 and k for the sequence from Figure 7: (a=1.165; φ=75°; sequence of the first- order).

XS2: (146027.769, 146027.768, 146027.753, 146027.757, 146027.753…); k: (228964.474, 228964.471, 228964.421, 228964.418, 228964.419…). (b= 200, that's why the numbers are so big).

Let us present similar values for the second- order sequence: (a=1.565; j=31o).
XS2: (15392.2605, 15392.2604, 15392.2600, 15392.2601, 15392.2611 …); k: (717.8548, 717.8555, 717.8555, 717.8553, 717.8553…)

This means that formula **FN2** is correct and there can be no doubt about it!

To use **FN2** in practice, we need to learn how to do the inverse transformation: from X-large ones we get (x,y) – small ones, i.e. coordinates on the ellipse.

This problem can be solved using a standard method, using tables, since the correspondence function between X and (x,y) is smooth. Let's see what the correspondence between the points of the ellipse and the points X looks like for a specific ellipse (a=1.165) at the different normal angles φ. See Figure 8.

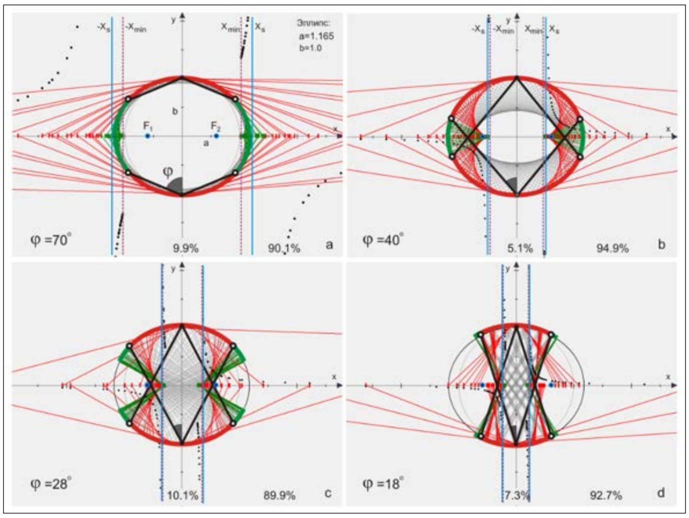

Figure 8: An image of 120 reflections in the ellipse with their one-dimensional representation at four different initial (normal) angles j. For a given ellipse, y4=30.87° and yF=49.36° (see Figure 3), that is the angle y4 is between Figure 8a and 8b, and the angle yF is between Figure 8b and 8c. The percentages below are the "probability" of being in the "green" or "red" zone (the percentage of the corresponding events out of the total).

The points of the ellipses along the perimeter are painted in different colors.

Those regions whose one-dimensional representations lie within the interval [$-X_S, X_S$] are outlined in green; those points of the ellipse whose X lie outside [$-X_S, X_S$] are circled in red; those regions where there are no reflection points (Figure 8c, 8d) are not circled at all.

Let's imagine a cartoon in which the angle φ continuously changes from 90° to 0° and in Figure 8 shows frames from it. What will we see?

Initially, the entire ellipse is green. With the beginning of the cartoon, the red zone appears and begins to quickly increase, "eating" the green one... and at the moment of the angle φ=ψ4 "eating up" it completely! The entire ellipse turns red. (ψ4 is the angle at which the coefficient *k* in **FN2** becomes equal to zero and our plot "turns over" around the X axis).

After the angle ψ4 (with a further decrease of the angle φ), the green zone appears again – according to the "exclusive OR" principle – and begins to increase, "eating" already the red zone. This occurs until the second critical point - angle yF.

At this moment (yF is the boundary of the **FN1** signature change), a significant piece of the green zone simply disappears and becomes uncolored (the order of movement changes from 1st to 2nd). After this, both zones (green and red) quickly decrease to zero at point φ=0. (At the same time, the green zone, as it was – after ψF – remains very narrow, as well as the corresponding intervals [X_S, X_{min}], [$-X_{min}$, $-X_S$]).

By compiling the appropriate tables, you can obtain the inverse transformation X -> (x,y). Note that we cannot determine the Y coordinate signature from a one-dimensional representation, and our answer will always be: either (x,y) or (x,-y)!

Conclusion

Our situation is somewhat reminiscent of the John Horton Conway algorithm for determining the relative probability of a sequence of heads and tails [4]. This formula p=(AA-AB)/(BB-BA) was also be first proposed by John Horton Conway and then proven. (AB is an integer obtained from two sequences A and B of heads and tails using a simple procedure) [5].

But our formula **FN2** will apparently be much more difficult to prove. Which follows from a simple difference in notation. Compare the notation of Conway's algorithm and our multi-story construction for **Formula Number Two**.

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