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The Transport of Species of Structures along the Braid Group

Pemha Binyam Gabriel Cedric* and Ikollo Ndoumbe Moïse

Department of Mathematics and Computer Sciences, Faculty of Sciences,University of Douala, PO Box 24157, Douala, Cameroon

ABSTRACT

The purpose of this paper is to present in an introductory was the notion of transport of t-structures of a given species. The letter t symbolizes a braid with m strands that is performed on each element of $[U]_m$. The transport will therefore be unique on each element of $[U]_m$ up to isomorphism because a braid is an isotopy class. This paper contains the basic concepts of the combinatorial theory of species of t-structures. We begin with some general considerations on the notion of t-structure, everywhere present in mathematics and theoretical computer science. These preliminary considerations lead us in a natural manner to the fundamental concept of species of structures. The definition of species puts the emphasis on the transport of t-structures along bijections of $\mathfrak{B}_m$.

***Corresponding author**

Pemha Binyam Gabriel Cedric, Department of Mathematics and Computer Sciences, Faculty of Sciences,University of Douala, PO Box 24157, Douala, Cameroon.

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Introduction

The combinatorial theory of species, introduced by Joyal in , in this general framework [1]. Initially, Joyal considers a species of structures as a functor between two categories. F. Bergeron for his part, puts the emphasis on the transport along the bijection $\mathfrak{S}_m$ [2]. Here, we will emphasize the transport of species of structures along the braids group $\mathfrak{B}_m$ [3-7]. To do this, we will need a set $[U]_m$, a set with $card(U)$ elements, U being a set such that each of its elements has m strands, m≥ 2.

The particularity in this new way of thinking is to generate even more combinatorial structures of species [8,9]. We use the braids group with m strands that we will act on the set of t -structures of a species $\mathfrak{F}$. The action in question will have effects on the labels, this means that the vertices of a t -structure $\mathfrak{S}_t$ will be renamed by braid f_i-words obtained from the strands which are inside them.

It provides a unified understanding of the use of generating series for both labeled and unlabeled structures, as well as a tool for the specification and analysis of these structures [10,11]. Of particular importance is its capacity to transform recursive definitions of structures into functional or differential equations, and conversely. Encompassing the description of structures together with permutation group actions, the theory of species conciliates the calculus of generating series and functional equations with Pólya theory, following previous efforts to establish an algebra of cycle index series.

We start with some considerations about the notion of t-structure. This approach, newly introduced here, will not only allow us, through preliminary considerations, to better understand in a natural way the fundamental concept of the species of t-structures. But also to undertake even larger lodges than those we had when we was just talking about species structures [12,13].

Section 2 presents first, notion of t-structure and fundamental example and then presents a study of algebraic structure of the group $\mathfrak{B}_m$. Section 3 is devoted to transport of species of t-structures.

Preliminaries

Notion of t-structures

Definition 0.1: A t-structure S_t is a construction γ_t which one performs on a finite set $[U]_m$;

where $[U]_m$ is a finite set of Card (U) elements, in which each of its elements has m strands.

It consists of a pair

$$s_t = (\gamma_t, [U]_m)$$

It is customary to say that $[U]_m$ is the underlying set of the structure S_t or even that S_t is a structure constructed from the set $[U]_m$.

Example 0.1: Let $\mathfrak{F}$ be a species of t-structures on $[U]_3$, where $U = \{1,2,3\}$.

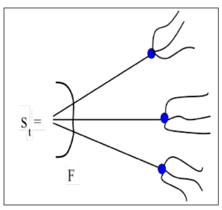

Figure 1: A combinatorial t-structure of species $\mathfrak{F}$

F.Bergeron defines the transport of species of structures as the action of the symmetric group $\mathfrak{S}_m$ on $\mathfrak{F}[U]$ the set of t-structures on U, and yet we want to define the action of the braid group $\mathfrak{B}_m$ on $\mathfrak{F}[U]$: Hence the name, the transport of species of -structures [14].

Algebraic Structure of Group $\mathfrak{B}_m$

Originally, the m-strand braid group $\mathfrak{B}_m$ is defined as the group of isotopy classes of geometric braids with m strands [15,3]. An algebraic exposition has been established by E. Artin, and it is this aspect that will be used in this work [1]. For us, $\mathfrak{B}_m$ is therefore the group presented as follows:

$$\mathfrak{B}_m =< \sigma_1,\cdots,\sigma_{m-1} \begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & for\, |i-j| \geq 2; \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & for\, |i-j| = 1. \end{cases} > \quad (1)$$

Thus a braid with strands is an equivalence class of words in the letters $\sigma_i^{f_i}$, where the map $f_i:[m]\to\{-1,1\}$ is defined by:

$$f_i = \begin{cases} 1, & if\ the\ i^{th}\ strand\ passes\ below\ the\ (1+i)^{th}\ strand\ ; \\ -1, & if\ the\ i^{th}\ strand\ passes\ above\ the\ (1+i)^{th}\ strand. \end{cases}$$

Such words will be called f_i-braid words. We will refer to the letters σ_i as generators of Artin. The standard correspondence between elements of $\mathfrak{B}_m$'s presentation and geometric braids is to use $\sigma_i^{f_i}$ as a code for the geometric braid where the i^{th} strand and the $(1+i)^{th}$ strand cross, with the convention that the strand originally in position $(1+i)$ passes above (resp. below) the other if f_i is counted positively (resp. negatively).

The diagrammatic interpretations of the letters $\sigma_i^{f_i}$ illustrate this fact.

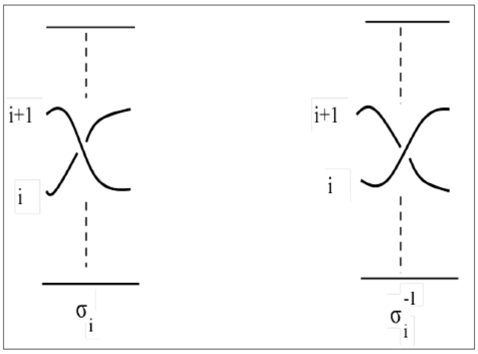

Figure 2: Interpretation of the letters $\sigma_i^{f_i}$ as a diagram

Notation 0.1: Let $\sum_m$ denote the alphabet whose letters are the $\sigma_i^{f_i}$. The set $\{\sigma_i^{f_i}; i\in[m], f_i\in\{-1,1\}\}$ and $\sim$ the group congruence generated by the relations of the presentation (1). $\sim$ is defined by:

$$\sigma_i^{f_i}\sigma_j^{f_j} = \sigma_j^{f_j}\sigma_i^{f_i}\ if\ |i-j|>1$$

and

$$\sigma_i^{f_i}\sigma_j^{f_j}\sigma_i^{f_i} = \sigma_j^{f_j}\sigma_i^{f_i}\sigma_j^{f_j}\ if\ |i-j|=1$$

Definition 0.2: A f_i-word of braid $\omega_t^{(i)}$ is a concatenation of the letters $\sigma_i^{f_i}$, $i\in[m]$, obtained by elementary crossing of the i^{th} strand with the remaining strands.

$$\omega_t^{(i)} = \prod_{i\in\mathcal{I}} \sigma_i^{f_i}$$

Where $\mathcal{I}$ denotes the set of possible elementary different f_i-crossings of the i^{th} strand with the remaining strands.

If f_i, $\forall i\in[m]$, is counted positively (resp. negatively) we say that $\omega_t^{(i)}$ is a positive (resp. negative) braid word. Thus any braid ω_t can be seen as an equivalence class of $\sum_m$-words for the relation $\sim$. But counting of $\sum_m$-words in their respective orbits will be discussed in a future paper. Here is a diagram interpretation illustrating a geometric braid.

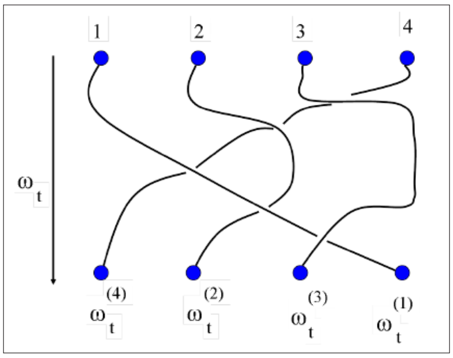

Figure 3: Interpreting braid f_i-words as a diagram

$$\omega_t^{(4)} = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1},\ \omega_t^{(2)} = \sigma_2^{-1}\sigma_1^{-1},\ \omega_t^{(3)} = \sigma_3^{-1}\sigma_1,\ \omega_t^{(1)} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1.$$

$\omega_t\in\mathfrak{B}_m$ is a bijection such that $\omega_t:[U]_m\to[V]_m$ is given by the formula $\omega_t(i) = \omega_t^{(i)}\in[V]_m$; more simply

$$\omega_t = \prod_{i\in[U]_m}\omega_t^{(i)}.$$

Transport of Species of t-Structures

To better understand the notion of transport of t-Structures s_t, consider the following proposition.

Proposition 0.1: Let s_t be a t-structure on $[U]_m$, the transport of the t-Structures s_t along ω_t gives on arrival a t-structure on $[V]_m$.

Proof 0.1: $s_t = (\gamma_t, [U]_m)$ is a construction γ_t on $[U]_m$ and ω_t, a f_i-word concatenation of braid $\omega_t^{(i)}$, $i\in[U]_m$. During transport along ω_t, we observe that each element $u\in[U]_m$ is send on a braid f_u-word $\omega_t^{(u)}$. As a result, every element of $u\in[U]_m$ is relabeled by a braid f_u-word $\omega_t^{(u)}\in[V]_m$. This transport therefore preserves the t-structure while relabeling the vertices of this t-structure.

Remark 0.1: Obviously $Card([U]_m) = Card([V]_m)$, however the construction of the elements of $[V]_m$ is not unique along ω_t. For each vertex of $[V]_m$, we have $2^{m-1}(m-1)!$ possible braid f_i- words.

Example 0.2: Take as an example the transport of the t-structures of the species of rooted trees, a, along ω_t. The species we use here are those introduced by A. Joyal [14].

For each finite set $[U]_m$, we denote by $a[U]_m$ the set of all t-structures of species of rooted trees on $[U]_m$. Thus, if $\mathfrak{g}[U]_m$ denotes the set of all structures of simple graph on $[U]_m$, $\mathfrak{g}$. That means, $\mathfrak{g}[U]_m = \{g_t \mid g_t = (\gamma_t,[U]_m), \gamma_t\in\mathcal{P}^{[2]}[U]_m\}$

where $\mathcal{P}^{[2]}[U]_m$ stands for the collection of (unordered) pairs of elements of $[U]_m$. Then $\mathfrak{a}[U]_m = g^{\bullet}[U]_m$.

Moreover, each $\omega_t \in \mathfrak{B}_m$ induces, by transport of t-structures, a function

$$\mathfrak{a}[\omega_t]: a[U]_m \to a[V]_m$$

describing the t-transport of rooted trees along ω_t.

Formally, if $\mathfrak{a}_t = (\gamma_t, [U]_m) \in \mathfrak{a}[U]_m$, then

$$\mathfrak{a}[\omega_t](\mathfrak{a}_t) = \omega_t \cdot \mathfrak{a}_t = (\omega_t \cdot \gamma_t, [\ \ ]_m),$$

where $\omega_t \cdot \gamma_t$ is the set of pairs $\{\omega_t^{(u)}, \omega_t^{(v)}\}$, $u,v \in [U]_m$. Thus each edge $\{u,v\}$ of t-structures $\mathfrak{a}_t$ finds itself relabeled $\{\omega_t^{(u)}, \omega_t^{(v)}\}$ in $\omega_t \cdot \gamma_t$.

Since this transport of t-structures of rooted trees $\mathfrak{a}_t$ along ω_t is only a relabeling of the vertices and edges by ω_t. It is clear that for bijections $\omega_t : [U]_m \to [V]_m$ and $\varpi_t : [V]_m \to [W]_m$, one has:

$$\mathfrak{a}[\omega_t \circ \varpi_t] = \mathfrak{a}[\omega_t] \circ \mathfrak{a}[\varpi_t]$$

and that, for the identity map $Id_{[U]_m} : [U]_m \to [U]_m$ one has:

$$\mathfrak{a}\left[Id_{[U]_m}\right] = Id_{\mathfrak{a}[U]_m} .$$

These two equalities express the functoriality of the transports of t-structures $\mathfrak{a}[\omega_t]$. It is this property which is abstracted in the definition of species of t-structures.

Definition 0.3: A species of t-structures is a rule $\mathfrak{F}$ which produces:

1. for each finite set $[U]_m$, a finite set $\mathfrak{F}[U]_m$,
2. for each bijection $\omega_t : [U]_m \to [V]_m$, a function

$$\mathfrak{F}[\omega_t] : \mathfrak{F}[U]_m \to \mathfrak{F}[V]_m$$

The functions $\mathfrak{F}[\omega_t]$ should further satisfy the following functorial properties:

a) for all bijections $\omega_t : [U]_m \to [V]_m$ and $\varpi_t : [V]_m \to [W]_m$:
$\mathfrak{F}[\omega_t \circ \varpi_t] = \mathfrak{F}[\omega_t] \circ \mathfrak{F}[\varpi_t]$

b) for the identity map $Id_{[U]_m} : [U]_m \to [U]_m$: $\mathfrak{F}\left[Id_{[U]_m}\right] = Id_{\mathfrak{F}[U]_m}$.

Remark 0.2: An element $s_t \in \mathfrak{F}[U]_m$ is called a -structure of species $\mathfrak{F}$ on $[U]_m$; $[U]_m$ being a set of cardinality Card (U) and each element has m strand. The function $\mathfrak{F}[\omega_t]$ is called the transport of $\mathfrak{F}$ - t-structures along ω_t. The advantage of this definition of species is that the rule $\mathfrak{F}$, which produces the structures $\mathfrak{F}[U]_m$ and the transport functions $\mathfrak{F}[\omega_t]$, can be described in any fashion provided that the functoriality conditions hold.

Example 0.3: Reconsider the species of rooted tree $\mathfrak{a}_t = (\gamma_t, [U]_m)$, whose underlying set is U= {1,2,3,4} = [4]. Replace each element of $[U]_m$ by those of $[V]_m = \{\omega_t^{(1)}, \omega_t^{(2)}, \omega_t^{(3)}, \omega_t^{(4)}\}_m$ via the bijection ω_t described by the following figure. This figure clearly shows how the bijection ω_t described by the following figure. This figure clearly shows how the bijection ω_t allows the -transport of the rooted tree onto a corresponding rooted tree $\mathfrak{a}'$, on the set $[V]_m$. Simply by replacing each vertex $u \in [4]_m$ by the corresponding vertex $\omega_t \in [V]_m$. We say that the rooted tree a' has been obtained by transporting the rooted tree a along the bijection ω_t and we write

$$\mathfrak{a}' := \omega_t \cdot \mathfrak{a} ,$$

one has the following figure:

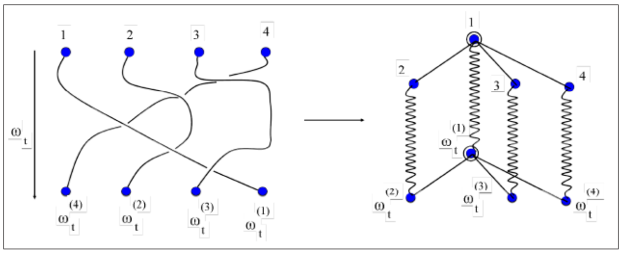

Figure 4: Labelled t-structures along ω_t

In this case, $\omega_t^{(4)} = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$, $\omega_t^{(2)} = \sigma_2^{-1}\sigma_1^{-1}$, $\omega_t^{(3)} = \sigma_3^{-1}\sigma_1$

$\omega_t^{(1)} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1$

From a purely set theoretical point of view, this amounts to replacing simultaneously each element $u \in [U]_m$ appearing in γ_t by the corresponding element $\omega_t(u)$ of $[V]_m$ in the expression of ω_t.

Definition 0.4: Consider two $\mathfrak{F}$ - t-structures $s_t \in \mathfrak{F}[U]_m$ and $\tau_t \in \mathfrak{F}[V]_m$. A bijection $\omega_t : [U]_m \to [V]_m$ is called an isomorphism of s_t to τ_t if $\tau_t = \mathfrak{F}[\omega_t](s_t)$. One says that these t-structures have the same isomorphism type.

Remark 0.3: An isomorphism from s_t to s_t is called be an automorphism of s_t . What is remarkable in the definition (0.3) is that the rule $\mathfrak{F}$ which generates the sets $\mathfrak{F}[U]_m$ and the transport $\mathfrak{F}[\omega_t]$. can be described arbitrarily provided the transport is consistent, verifying functoriality properties.

Theorem 0.1: Two $\mathfrak{F}$ -t-structures $s_t \in \mathfrak{F}[U]_m$ nd $\tau_t \in \mathfrak{F}[V]_m$ are isomorphic t-structures if the elements of the underlying set are indistinguishable points.

Proof 0.2: Let s_t and τ_t be two isomorphic $\mathfrak{F}$ -t -structures belonging respectively to $\mathfrak{F}[U]_m$ When transporting from t-structure s_t to -structure τ_t, the labels of t-structure s_t are replaced to obtain those of t-structure τ_t on $[V]_m$. Similarly, the transport from t-structure τ_t to t-structure s_t cause the labels of t-structure τ_t to be replaced to obtain those of -structure s_t. This reciprocal transport $\tau_t = \mathfrak{F}[\omega_t](s_t)$ and $s_t = \mathfrak{F}[\omega_t](\tau_t)$,from .t .-structure s_t to t- structure τ_t is only possible when the two -structures have the same labels. That means they cannot be distinguished, they have the same type.

Conclusion

In this paper, it was question to present a species of structures along the braid group $\mathfrak{B}_m$.

It appears that, contrary to the symmetric group $\mathfrak{S}_m$, the braid group $\mathfrak{B}_m$. constitutes a combinatorial enrichment of the notion of species of structures.

The transport of t-structures of a species $\mathfrak{F}$ consists of a relabeling of the underlying set $[U]_m$ along $\omega_t \in \mathfrak{B}_m$. The notion of t-structures preserves the form of the species of structures on a set of cardinality.

In the next paper, we will introduce some of the first power series that can be associated to species: generating series, types generating series, cycle index sees. They serve to encode all the information concerning labeled and unlabeled enumeration [16].

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