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UC Factorization and Inversion of Tridiagonal Matrices

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ABSTRACT

Tridiagonal matrix inversion using a UC factorization decomposition of classic tridiagonal matrices is presented in this paper. Further we provide two algorithms to measure the runtimes of both algorithms and how to validate the effectiveness of our decomposition method. All of this serves as foundation for a couple of points of interest.

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Introduction

Abstractions based on tridiagonal matrices are essential not just to linear algebra, but also to applied mathematics, for example, numerical analysis, applications in orthogonal polynomials, engineering, telecommunications system analysis, as well as system identification and signal processing, for instance speech decoding as well as deconvolution. They are also important in applications including special functions, partial differential equations, and of course, linear algebra itself [1-6].

The matrices that need to be inverted are tridiagonal and arise in various applications in these domains. In order to meet this need, multiple approaches have been suggested, such as parallel computations, faster algorithms indirect methods and even specific direct expressions for special cases [7-12]. In addition, many authors have focused on tridiagonal matrices of fixed dimension, including LU factorizations, determinants and inverses derivations for special cases of these matrices .

Over the last few years a lot of progress has been made with respect to the formulation and study of tridiagonal Toeplitz matrices. To enumerate few of such advancements, it could be referred to the works in which the researchers J. Jia, T. Sogabe, and M. El-Mikkawy presented an explicit formula of the tridiagonal Toeplitz matrix inverse [13]. They propose a unique manner to compute the inverse directly without going through those iterative processes or other advanced numerical methods.

Tanasescu and Popescu also designed a fast tridiagonal SVD algorithm in [14]. They proposed a method that accelerates the decomposition step by applying block diagonalization. Gérard Meurant has also provided an in-depth study on symmetric tridiagonal matrix inversion as well [1].

In this work we consider tridiagonal matrices of the form:

$$T=\begin{pmatrix} a_1 & c_1 & 0 & \cdots & 0 \\ b_1 & a_2 & c_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{pmatrix} \in \mathbb{M}_n(\mathbb{C}) \qquad (1)$$

Where $(a_i)_{1\le i\le n-1}$, $(b_i)_{1\le i\le n-1}$ and $(c_i)_{1\le i\le n-1}$ are sequences of real or complex numbers such that $c_i\neq 0$ for i=1,2,..,n-1.

We study a new decomposition method for tridiagonal matrices.

It uses a UC decomposition, with:

$$T=UC \qquad (2)$$

Where U is the upper triangular matrix, and C is the companion matrix.

The UC Factorization of a Tridiagonal Matrix

In this part, we provide a straightforward definition for the factorization of a tridiagonal matrix T
(1) and also outline the process for computing the matrices U and C.
The subsequent theorem presents a factorized formulation of the matrix T.

Theorem 1: Any tridiagonal matrix T is equal to the product T=UC, where:

$$C=\begin{pmatrix} 0 & & \cdots & & 0 & \alpha_1 \\ 1 & \ddots & & & \vdots & \frac{\alpha_2}{b_1} \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \frac{\alpha_n}{b_{n-1}} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$$

And

$$U = \begin{pmatrix} 1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & c_1 & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & b_{n-2} \\ & & & & \ddots & a_{n-1} \\ 0 & & \cdots & & 0 & c_{n-1} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$$

With:

$$\alpha_n = a_n \quad and \quad \alpha_r = (-1)^{n-r}\left(\prod_{k=r}^{n-1} c_k\right)^{-1}\Delta_r \quad where \ \ r = 1,2,3,...,n-1$$

And

$$\Delta_r = \det\begin{pmatrix} a_r & c_r & 0 & \cdots & 0 \\ b_r & a_{r+1} & c_{r+1} & \ddots & \vdots \\ 0 & b_{r+1} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{pmatrix}.$$

Proof 1: Certainly, we acquire $\alpha_n = a_n$, and

$$det(T) = \Delta_1 = (-1)^{n-1}\left(\prod_{k=1}^{n-1} c_k\right)x_1$$

And next, from the direct product of matrices *U* and *C*, we show the following recurrence relations:

$$\alpha_{n-1} + \frac{\alpha_n a_{n-1}}{c_{n-1}} = b_{n-1}$$

$$\alpha_r + \frac{\alpha_{r+1}a_r}{c_r} + \frac{\alpha_{r+2}b_r}{c_{r+1}} = 0 \ for \ r = 1,2,...,n-2$$

We receive:

$$\alpha_{n-1} = \frac{1}{c_{n-1}}(b_{n-1}c_{n-1} - a_n a_{n-1})$$

$$= -\frac{1}{c_{n-1}}\Delta_{n-1}$$

And

$$\alpha_{n-2} = -\left(\frac{\alpha_{n-1}}{c_{n-2}}a_{n-2} + \frac{\alpha_n}{c_{n-1}}b_{n-2}\right)$$

$$= \frac{1}{c_{n-2}c_{n-1}}(a_{n-2}\Delta_{n-1} + a_n b_{n-2}c_{n-2})$$

$$= \frac{1}{c_{n-2}c_{n-1}}(a_{n-2}\Delta_{n-1} + a_n b_{n-2}c_{n-2})$$

$$= \frac{1}{c_{n-2}c_{n-1}}(a_{n-2}\Delta_{n-2})$$

We arrive at the expression for α_r, so appealing to the inductive process, the next case is for *r*=1, and so we then need to prove the equation. In particular, we can write it as $\alpha_1 + \frac{\alpha_2 a_1}{c_1} + \frac{\alpha_3 b_1}{c_2} = 0$.

$$\alpha_1 = -\left(\frac{\alpha_2 a_1}{c_1} + \frac{\alpha_3 b_1}{c_2}\right)$$

$$= -\left(\frac{-1}{c_1 c_2}(\alpha_2 a_1 c_2 + \alpha_3 c_1 b_1)\right)$$

$$= -\frac{1}{c_1c_2}\left((-1)^{n-2}a_1\left(\prod_{k=2}^{n-1} c_k\right)^{-1}\Delta_2 + (-1)^{n-3}c_1 b_1\left(\prod_{k=3}^{n-1} c_k\right)^{-1}\Delta_3\right)$$

$$= -(-1)^{n-2}\left(\prod_{k=1}^{n-1} c_k\right)^{-1}(a_1\Delta_2 - c_1 b_1\Delta_3)$$

$$= (-1)^{n-1}\left(\prod_{k=1}^{n-1} c_k\right)^{-1}\Delta_1$$

Theorem 2: We define U as upper triangular matrix, then assuming U to be non-singular, we can write:

$$U = U_1U_2 \qquad (3)$$

Where U_1 and U_2 are defined as:

$$U_1 = \begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_n \\ 0 & & \cdots & 0 & 1 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & \gamma_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & & \cdots & 0 & 1 \end{pmatrix}$$

Where:

$$\beta_1 = a_1, \ \beta_i = a_i - \frac{b_i}{\beta_{i-1}}, \ for \ i = 1...n$$

$$\gamma_1 = \frac{b_1}{a_1}, \ \gamma_i = \frac{b_i}{a_i - \beta_{i-1}}, \ for \ i = 1...n$$

Proof 2: Using (3), we obtain the following relations:

$$a_n = \gamma_n + \beta_n, \ n \in 1,...,N$$

$$b_n = \gamma_{n-1} + \gamma_{n-3}, \ n \in 2,...,N$$

We can solve these equations recursively as follows:

$$\beta_1 = a_1 - \gamma_1 \quad \gamma_2 = \frac{b_1}{a_1 - \gamma_1}$$

$$\beta_2 = a_2 - \gamma_2 \quad \gamma_3 = \frac{b_1}{a_1 - \gamma_2}$$

Continued this way and using induction we arrive at the conclusion.

Inverse of Tridiagonal Matrix (Algorithm 1):
In this section we declare the expression of companion matrix inverse.

Lemma 1: Considering the companion matrix:

$$C = \begin{pmatrix} 0 & & \cdots & & 0 & x_1 \\ 1 & \ddots & & & \vdots & \frac{x_2}{b_1} \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \frac{x_n}{b_{n-1}} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C})$$

be a companion matrix, then det(C)=$(-1)^{n-1}$ x_1 and:

$$C^{-1} = \begin{pmatrix} -\frac{x_2}{x_1 c_1} & 1 & 0 & \cdots & 0 \\ & 0 & & & 0 \\ \vdots & & & \ddots & \vdots \\ & & & \ddots & \vdots \\ -\frac{x_n}{x_1 c_{n-1}} & & & & 1 \\ 1 & 0 & & \cdots & 0 \end{pmatrix}$$

Proof 3: The identity C^{-1} C=I_n, where I_n denote the n×n identity, is easily verifiable matrix.

Theorem 3: T is a tridiagonal matrix, and T=U_1 U_2 C then we can write:

$$T^{-1} = C^{-1} U_2^{-1} U_1^{-1}$$

Inverse of a Toeplitz Tridiagonal Matrix (Algoritm 2):
Within this part, we provide recurrence formulas for the columns of the inverse of a tridiagonal matrix.

Suppose T is non-singular and designate:

$$T^{-1} = (C_1, ..., C_n),$$

Where C_j is the j[th] column inverse of T^{-1}.

From the relationship $T^{-1}T = I_n$, where I_n represents the identity matrix of order n, we derive
the following expressions:

$$C_{n-1} = \frac{1}{c_{n-1}}(E_n - a_n C_n)$$

$$C_{j-1} = \frac{1}{c_{j-1}}(E_j - a_j C_j - b_{j+1} C_{j+1}) \ for \ j = n-1, ..., 2.$$

Where $E_j = [(\delta)_{1 \leq i,j \leq n}]^t \in \mathbb{K}^n$.

We provide simple recurrence formulas to calculate the terms of the sequence denoted as c_n.
Considering the sequence of numbers $(A_i)_{0 \leq i \leq n}$ defined as:

$$A_0 = 1$$

$$a_1 A_0 + c_1 A_1 = 0$$

And

$$b_{i+1} A_{i-1} + a_{i+1} A_i + c_{i+1} A_{i+1} = 0 \ for \ 1 \leq i \leq n-1.$$

Theorem 4: Suppose that $A_n \neq 0$, then T is invertible and if $T^{-1}=(C_1,...,C_n)$ is the inverse of T then:

$$C_n = [\frac{-A_0}{A_n}, ..., \frac{-A_{n-1}}{n}]^t$$

Proof 4: $det(T) = \Delta_n = (-1)^n (\prod_{k-1}^{n} c_k) A_n \neq 0,$, the matrix T is invertible

We have $TC_n=E_n$, then the proof is completed.

Numerical Results
In this section, we give a numerical example to illustrate the effectiveness of our algorithm.
Our algorithm is tested by MATLAB R2014a.

Consider the following 5-by-5 tridiagonal matrix

$$T = \begin{pmatrix} 5/2i+1 & -5i & 0 & 0 & 0 \\ -9/7+i & 5/2i+1 & -5i & 0 & 0 \\ 0 & -9/7+i & 5/2i+1 & -5i & 0 \\ 0 & 0 & -9/7+i & 5/2i+1 & -5i \\ 0 & 0 & 0 & -9/7+i & 5/2i+1 \end{pmatrix}$$

Therefore the inverse of the matrix T is given as:

$$T^{-1} = \begin{pmatrix} -0.0279-0.2626i & -0.0831+0.0928i & 0.0487+0.0042i & -0.0191-0.0131i & -0.0044+0.0133i \\ -0.0547-0.3011i & -0.1094-0.1745i & -0.0289+0.0755i & 0.0295-0.0244i & -0.0191-0.0131i \\ -0.0479-0.2909i & -0.1027-0.1698i & -0.0251-0.1686i & -0.0289+0.0755i & 0.0487+0.0042i \\ -0.1821-0.2909i & -0.1865-0.1363i & -0.1027-0.1698i & -0.1094-0.1745i & -0.0831+0.0928i \\ -0.0906-0.5017i & -0.1821-0.2909i & -0.0479-0.2909i & -0.0547-0.3011i & -0.0279-0.2626i \end{pmatrix}$$

In the table we give a comparison of the running time between this algorithm and LU method in MATLAB R2014a

The running time (in seconds) of two algorithms in MATLAB R2014a.

Table 1: The Running Time

Size of the matrix (n)	Algorithm 1	Algorithm 2	LU method
100	0.042833	0.003815	0.865027
200	0.066880	0.007520	3.075030
300	0.099705	0.019013	6.978036
500	0.174550	0.059326	24.863873
1000	0.537444	0.198082	232.363131

Conclusion
Our research has yielded intriguing outcomes by employing the UC decomposition, with C as a companion matrix and U as an upper triangular matrix. Furthermore, we've innovated a fresh algorithm designed specifically for computing the inverse of tridiagonal matrices [15-22].

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